Avalanche Statistics in Disordered Visco-Elastic Interfaces

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(Conventional) Depinning of an interface (1)

- Out-of-equilibrium: slow Driving \( w(t) = V_0 t \)
- Disorder: random distribution \( f_{i}^{\text{dis}}(h_i) \)
- Competition: Disorder VS Elasticity

\[
\eta \partial_t h_i = k_0 (w - h_i) + f_{i}^{\text{dis}}(h_i) + k_1 \nabla^2 h_i
\]

Two time scales, \( \eta \ll dh/V_0 \)
(Conventional) Depinning of an interface (2)

- ⇒ scale-free distribution of avalanches
  \[ P(S) \sim S^{-\tau} e^{-S/S_m} \]
- ⇒ no correlations
- ⇒ Stationary stress \( \sigma \equiv k_0(w - h) \)

Works for: cracks propagation, magnetization domains, etc.
Correlations?

- compression of solids, earthquakes dynamics: correlations!

low strain rate: $\gamma = 10^{-6} \text{ s}^{-1}$

Micro-crystals compression
(S. Papanikolaou et al., Nature 490, 51721 (2012)).

- How to get correlations in Depinning Framework?
Simple Model of Viscoelastic Material

One degree of freedom $h_i$ per site $\rightarrow$ Two degrees of freedom $h_i, \phi_i$ per site

Purely elastic $\rightarrow$ visco-elastic

$$F_{h_i \rightarrow h_{i+1}} = k(h_{i+1} - h_i) \rightarrow F_{\phi \rightarrow h} = \eta u \partial_t (h - \phi) + \text{elastic int.}$$
Visco-Elastic Depinning: Definition

\[ \eta \partial_t h_i = k_0(w - h_i) + f_{i}^{\text{dis}}(h_i) + k_1 \nabla^2 h_i + k_2(\nabla^2 h_i - u_i) \]

\[ \eta_u \partial_t u_i = k_2(\nabla^2 h_i - u_i), \]

with \( u_i \equiv \phi_i - h_i + h_{i-1} - \phi_{i-1} \)
Numerical Results, Two Dimensions

Visco-elastic model: Aftershocks!

Also:
Avalanche distribution: \( P(S) \sim S^{-\tau}, \tau \in [1.7, 1.8], \) realistic for Earthquakes.
Numerical Results, Mean Field

- Strong correlations appear: aftershocks, etc.
- Global avalanches, periodically.

(S. Papanikolaou et al., Nature 490, 51721 (2012)).
Approximation: Identical Wells

- White noise:


\[ E_{i}^{\text{dis}} \]

\[ h_{i} \]

\[ \text{width} \rightarrow 0 \]

\[ \mathbb{P}(z) = g(z) \]

- Simplify more: same depth for all wells, \( f_{i}^{\text{th}}(h_{i}) \equiv f^{\text{th}} \equiv \text{const.} \)

\[ E_{i}^{\text{dis}} \]

\[ h_{i} \]

\[ \text{width} \rightarrow 0 \]

\[ \mathbb{P}(z) = g(z) \]
Mean Field: conventional Depinning

\[ \delta_i \equiv \text{how much force before getting out of pinning well} \]

\[ \delta_i = f^{th} - k_0 (w - h_i) - k_1 (\bar{h} - h_i) \]

\[ P_w(\delta) \rightarrow P_{w+dw}(\delta) = ? \]

(0) Driving: shift in \( \delta \) by \( k_0 dw \): \( P(\delta) \leftarrow P(\delta + k_0 dw) \)

(1) fraction \( P(0) k_0 dw \) of system jumps from 0 to \( \delta = z (k_0 + k_1) \):
\[ P(\delta) d\delta \leftarrow P(\delta) d\delta + P(0) k_0 dw \cdot g(z) dz \]

(0) \( \oplus \) (1): \[ \frac{P_{\text{step}1}(\delta) - P_w(\delta)}{k_0 dw} = \frac{\partial P_w}{\partial \delta}(\delta) + P_w(0) \frac{g \left( \frac{\delta}{k_0+k_1} \right)}{k_0+k_1} \]

(2) jumps of \( z \Rightarrow \) increase in \( \bar{h} \): shift in \( \delta \) by \( P(0) \bar{z} k_1.k_0 dw \)
\[ \rightarrow \text{more steps (0), (1) } \rightarrow \text{(2): shift in } \delta \text{ by } \left( P(0) \bar{z} k_1 \right)^2.k_0 dw \]
\[ \rightarrow \text{more steps (0), (1) } \rightarrow \text{(2): ...} \]
\[ \ldots \text{ until shift } \approx 0. \]
Mean Field: conventional Depinning

\[ \delta_i \equiv \text{how much force before getting out of pinning well} \]

\[ \delta_i = f^{\text{th}} - k_0(w - h_i) - k_1(\bar{h} - h_i) \]

\[ P_w(\delta) \rightarrow P_{w+dW}(\delta) = ? \]

(0) Driving: shift in \( \delta \) by \( k_0dW \): \( P(\delta) \leftarrow P(\delta + k_0dW) \)

(1) fraction \( P(0)k_0dW \) of system jumps from 0 to \( \delta = z(k_0 + k_1) \):

\[ P(\delta)d\delta \leftarrow P(\delta)d\delta + P(0)k_0dW.g(z)dz \]

(0) \( \oplus \) (1):

\[ \frac{P_{\text{step}1}(\delta)-P_w(\delta)}{k_0dW} = \frac{\partial P_w}{\partial \delta}(\delta) + P_w(0)g\left(\frac{\delta}{k_0+k_1}\right) \]

(2) jumps of \( z \Rightarrow \) increase in \( \bar{h} \): shift in \( \delta \) by \( P(0)\bar{z}k_1.k_0dW \)

\[ \rightarrow \] more steps (0), (1) \( \rightarrow \) (2): shift in \( \delta \) by \( (P(0)\bar{z}k_1)^2.k_0dW \)

\[ \rightarrow \] more steps (0), (1) \( \rightarrow \) (2) : \[ \ldots \]

\[ \ldots \text{ until shift } \approx 0. \]
Mean Field: conventional Depinning

\( \delta_i \equiv \) how much force before getting out of pinning well

\[
\delta_i = f^{th} - k_0(w - h_i) - k_1(\bar{h} - h_i)
\]

\( P_w(\delta) \rightarrow P_{w+dw}(\delta) = ? \)

(0) Driving: shift in \( \delta \) by \( k_0dw \): \( P(\delta) \leftarrow P(\delta + k_0dw) \)

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\[
P(\delta)d\delta \leftarrow P(\delta)d\delta + P(0)k_0dw.g(z)dz
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(0) \( \oplus \) (1): \( \frac{P_{\text{step1}}(\delta) - P_w(\delta)}{k_0dw} = \frac{\partial P_w}{\partial \delta}(\delta) + P_w(0)\frac{g\left(\frac{\delta}{k_0+k_1}\right)}{k_0+k_1} \)

(2) jumps of \( z \Rightarrow \) increase in \( \bar{h} \): shift in \( \delta \) by \( P(0)\bar{z}k_1.k_0dw \)

\( \rightarrow \) more steps (0), (1) \( \rightarrow \) (2) : shift in \( \delta \) by \( (P(0)\bar{z}k_1)^2.k_0dw \)

\( \rightarrow \) more steps (0), (1) \( \rightarrow \) (2) : \( \ldots \)

\( \ldots \) until shift \( \approx 0 \).
Mean Field: conventional Depinning

$\delta_i \equiv$ how much force before getting out of pinning well

$\delta_i = f^\text{th} - k_0(w - h_i) - k_1(\bar{h} - h_i)$

$P_w(\delta) \rightarrow P_{w+dw}(\delta) = ?$

(0) Driving: shift in $\delta$ by $k_0dw$: $P(\delta) \leftarrow P(\delta + k_0dw)$

(1) fraction $P(0)k_0dw$ of system jumps from 0 to $\delta = z(k_0 + k_1)$:

$P(\delta)d\delta \leftarrow P(\delta)d\delta + P(0)k_0dw.g(z)dz$

(0) $\oplus$ (1): $\frac{P_{w+1}(\delta) - P_w(\delta)}{k_0dw} = \frac{\partial P_w}{\partial \delta}(\delta) + P_w(0)\frac{g\left(\frac{\delta}{k_0+k_1}\right)}{k_0+k_1}$

(2) jumps of $z \Rightarrow$ increase in $\bar{h}$: shift in $\delta$ by $P(0)\bar{z}k_1.k_0dw$

$\rightarrow$ more steps (0), (1) $\rightarrow$ (2) : shift in $\delta$ by $(P(0)\bar{z}k_1)^2.k_0dw$

$\rightarrow$ more steps (0), (1) $\rightarrow$ (2) : \ldots

\ldots until shift $\approx 0$. 
Mean Field: conventional Depinning

\[ \delta_i \equiv \text{how much force before getting out of pinning well} \]
\[ \delta_i = f^{\text{th}} - k_0(w - h_i) - k_1(\bar{h} - h_i) \]

\[ P_w(\delta) \to P_{w+dw}(\delta) = ? \]

(0) Driving: shift in \( \delta \) by \( k_0dw \): \( P(\delta) \leftarrow P(\delta + k_0dw) \)

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(0) \( \oplus \) (1) : \[ \frac{P_{\text{step1}}(\delta) - P_w(\delta)}{k_0dw} = \frac{\partial P_w}{\partial \delta}(\delta) + P_w(0) \frac{g\left(\frac{\delta}{k_0+k_1}\right)}{k_0+k_1} \]

(2) jumps of \( z \Rightarrow \text{increase in } \bar{h} \): shift in \( \delta \) by \( P(0)\bar{z}k_1.k_0dw \)
\to more steps (0), (1) \to (2) : shift in \( \delta \) by \( (P(0)\bar{z}k_1)^2.k_0dw \)
\to more steps (0), (1) \to (2) : \ldots
\ldots \text{ until shift } \approx 0. \]
Mean Field Solution

The stationary regime (fixed point) fulfils:

\[ 0 = \frac{\partial P}{\partial \delta} (\delta) + P(0) \frac{g(\frac{\delta}{k_0 + k_1})}{k_0 + k_1} \]  

(1)

Solution:

\[ P(\delta) = \frac{1 - \int_0^{\frac{\delta}{k_0 + k_1}} g(z)dz}{\overline{z}(k_0 + k_1)} \equiv Q(\delta, k_1) \]  

(2)
The stationary fixed point is given by $Q(\delta, k_1)$. The function $g(z) = e^{-z}$ is also shown in the diagram.
Mean Field Solution

From any initial state, fixed point reached in finite time.

If \( P(0) < \bar{z}k_1 \): finite avalanches with cutoff
\[
S_m = (1 - P(0)\bar{z}k_1)^{-2} = \left( \frac{k_0+k_1}{k_0} \right)^2
\]

If \( P(0) \geq \bar{z}k_1 \): divergent avalanches, more analysis needed.
Exact results: Mean Field (1)

Two limiting regimes compete ⇒ cycle emerges

\[ \delta_i = f^{th} - k_0(w - h_i) - k_1(h - h_i) - k_2(h - h_i - u_i) \] (3)
\[ \eta_u \partial_t u_i = k_2(h - h_i - u_i) \] (4)

\[ k_0 = 0.001, \quad k_1 = 0.1, \quad k_2 = 0.3 \]
Exact results: Mean Field (2)

Stress oscillates periodically:

\[ \sigma \text{ of } Q(k_1) \]

\[ \sigma \text{ of } Q(k_1 + k_2) \]
Conclusions

- understand aftershocks or avalanches’ correlations:
  **visco-elasticity** is a good candidate

- **microscopic** time $\eta_u \Rightarrow$ **emerging time scale** (periodicity)
  (Non-Equilibrium Non-Stationary State)

- **Fokker-Planck** analysis allows to understand the
  **time-dependent evolution** of the system state.

- periodic behaviour explained as a competition between
  (fast-dynamics) **stable** depinning critical point $Q(\delta, k_1 + k_2)$
  VS **unstable** one (but attractive for the slow dynamics)
  $Q(\delta, k_1)$
Thank You!
Three characteristic times

Three time scales: internal avalanche time, inter-aftershock time, driving

\[ \eta \ll \eta_u \ll \frac{dh}{V_0} \]

\[ \eta \partial_t h_i = k_0(w - h_i) + f_i^{\text{dis}}(h_i) + k_1 \nabla^2 h_i + k_2(\nabla^2 h_i - u_i) \]  
\[ \eta_u \partial_t u_i = k_2(\nabla^2 h_i - u_i), \]

- (i) During avalanches (time \( \sim \eta \))
  \( \Rightarrow \) dashpots are blocked, \( u_i \approx \text{const.} \) interface \( h \) jumps
  \( \rightarrow \) classical depinning evolution, with \( k_1^{\text{eff}} = k_1 + k_2 \)

- (ii) Between avalanches (time \( \sim \eta_u \))
  When \( h_i \) are pinned: relaxation of \( u_i \): \( u_i \rightarrow \nabla^2 h_i \).
  \( \Rightarrow \) triggers new avalanches (\( \sim \) aftershocks)
  \( \rightarrow \) drive towards classical depinning state, \( k_1^{\text{eff}} = k_1 \)

- (iii) When all \( h_i \) pinned and all \( u_i \) relaxed (time \( \sim \frac{dh}{V_0} \)):
  \( \Rightarrow \) drive, \( w \rightarrow w + dw \)
Mean Field: conventional Depinning (1) : Summary

- Fully-connected model: $(\nabla^2 h)_i \rightarrow \bar{h} - h_i$
- Equal pinning wells: $f_i^{\text{dis}}(h_i) \rightarrow f^{\text{th}}, P(z) = g(z)$
- Interface continuous motion discretized:
  \[
  \{ \partial_t h > 0 \Rightarrow \text{jump} \} \rightarrow \{ \delta_i < 0 \Rightarrow \text{jump} \}
  \]
  with \( \delta_i \equiv f^{\text{th}} - k_0(w - h_i) - k_1(\bar{h} - h_i) \)
- All sites equivalent: Fokker-Planck analysis
  \[
  \{ h_i, \forall i \} \rightarrow \{ \delta_i, \forall i \} \rightarrow P(\delta) \text{ describes whole system}
  \]
  \( N \) blocks \( \rightarrow \) \( N \) blocks \( \rightarrow \) \( \infty \) blocks
- \( \partial_t h = \cdots \rightarrow \partial_t P(\delta) = ?? \)
Numerical Results: 2D

- locally pseudo periodic
- local stress: oscillations between two values

Graph showing stress variation with w for patches A and B.