# Statistical mechanics of two-dimensional shuffled foams: prediction of the correlation between geometry and topology


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2. University of New Mexico, USA
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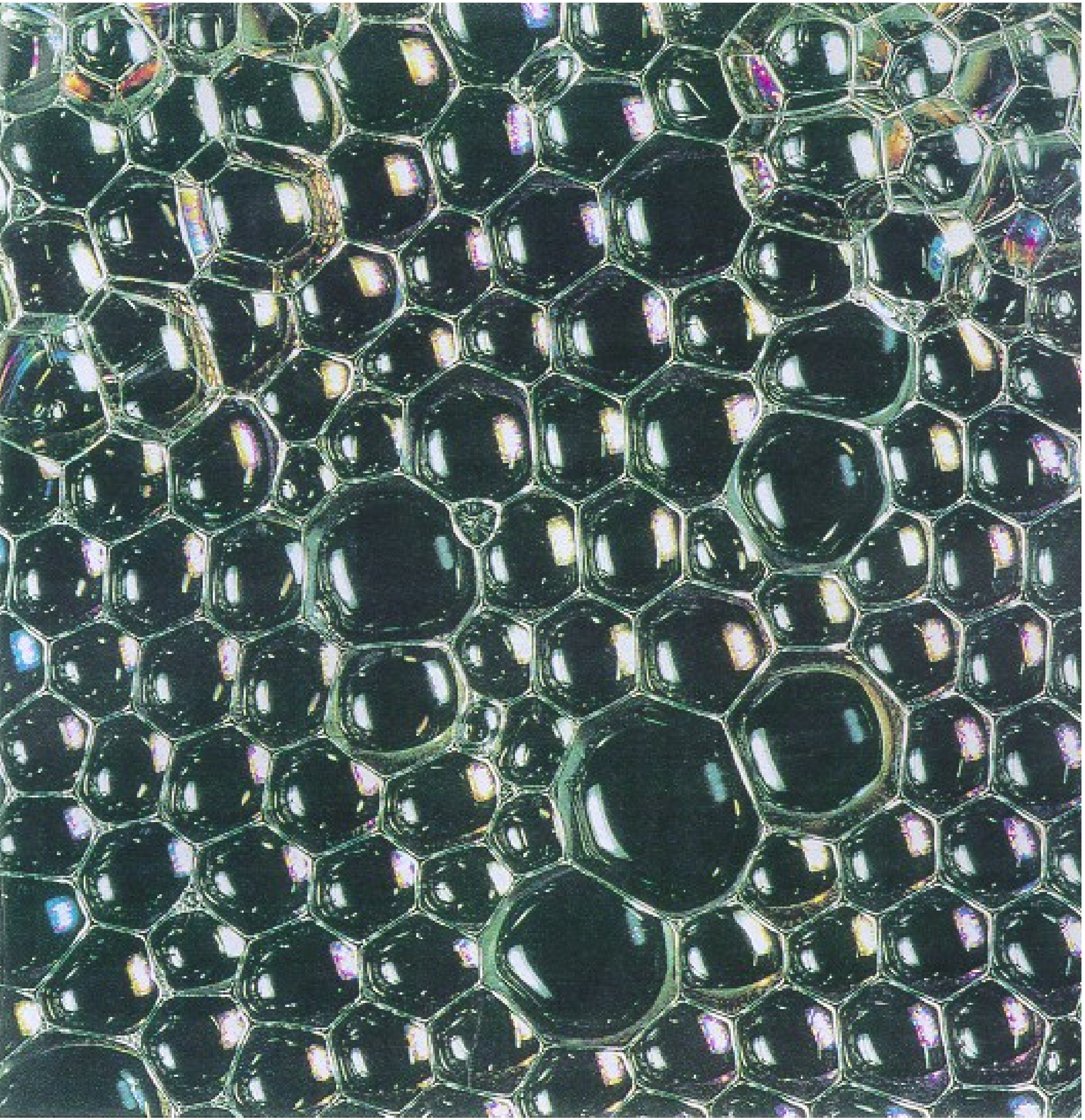
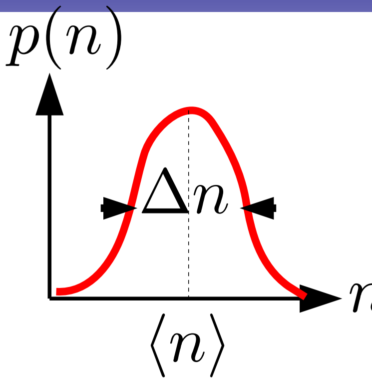
**Driven Disordered Systems, 05-06 June 2014, Grenoble**

# Disorders in foams

« Dry » foams = far beyond jamming transition !

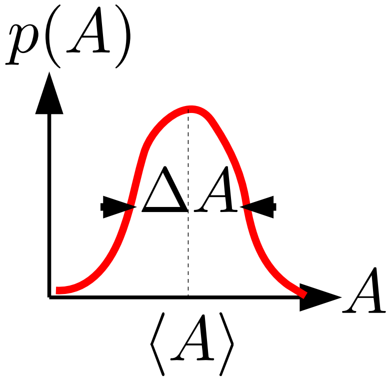


**- topological :**




$$\frac{\Delta n}{\langle n \rangle} = \frac{\sqrt{\langle n^2 \rangle - \langle n \rangle^2}}{\langle n \rangle}$$

**- geometrical :**

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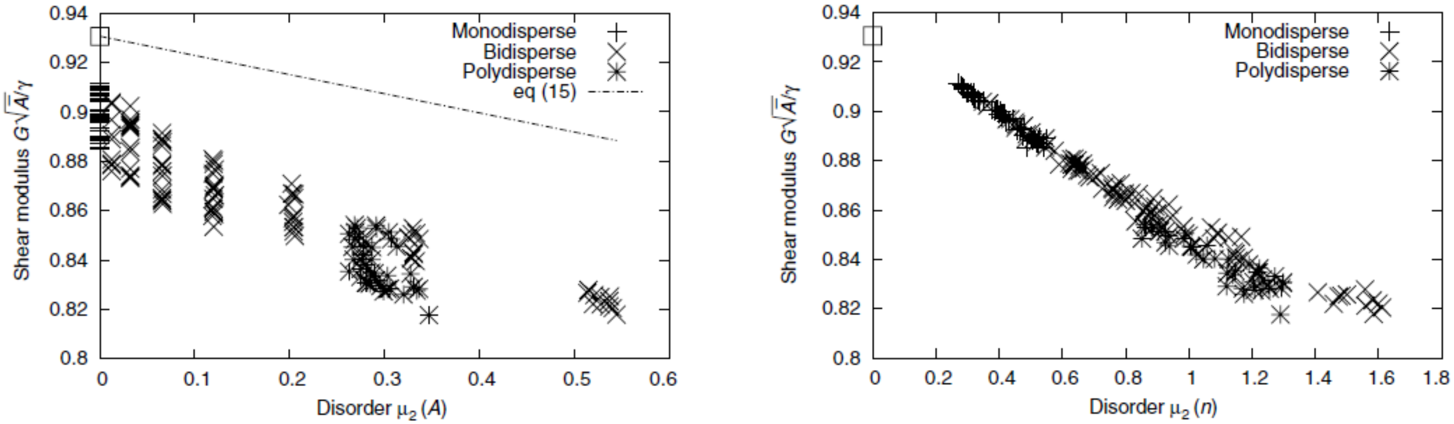
# Motivations

**- Mechanical properties of foams :**




*Cox & Whittick (2006)*

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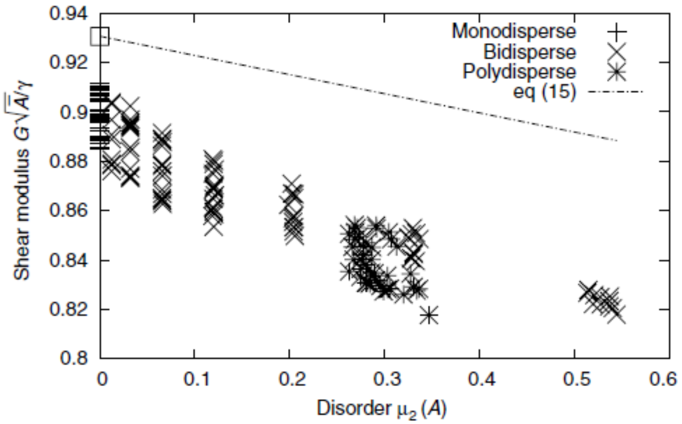


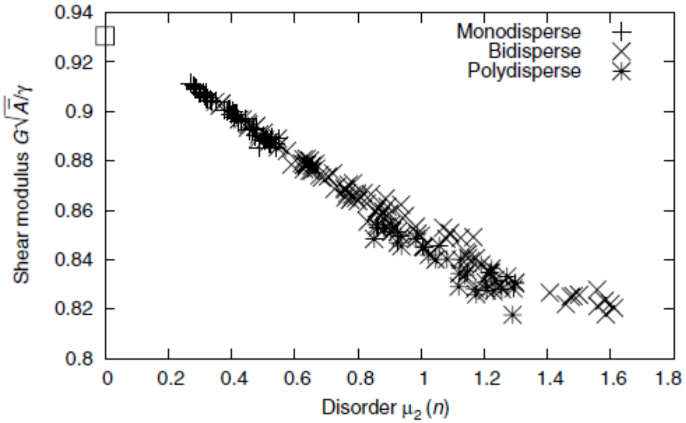


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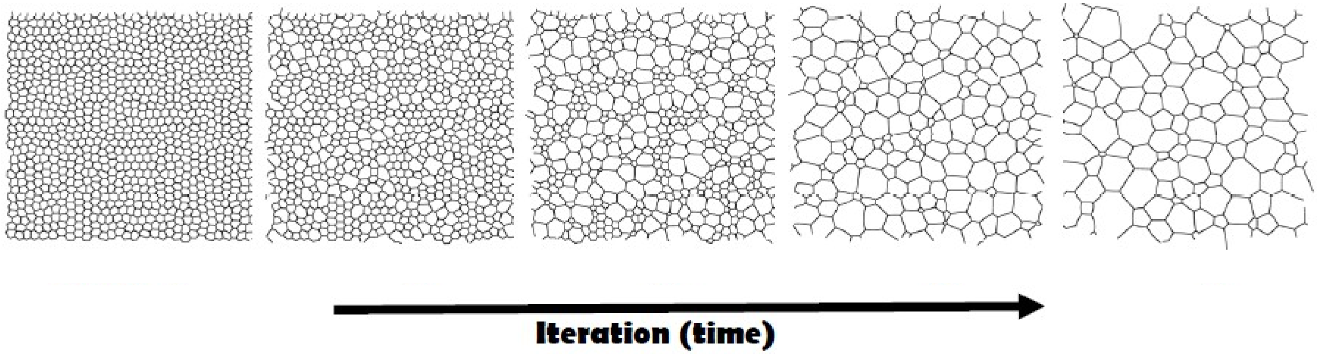
**- Coarsening rate :** depends on topological disorder



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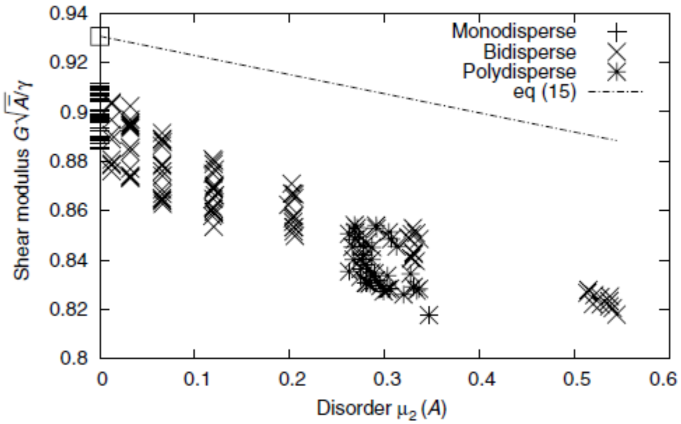


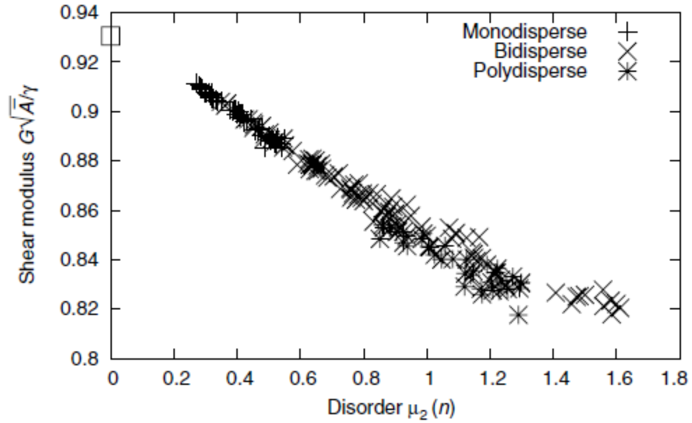


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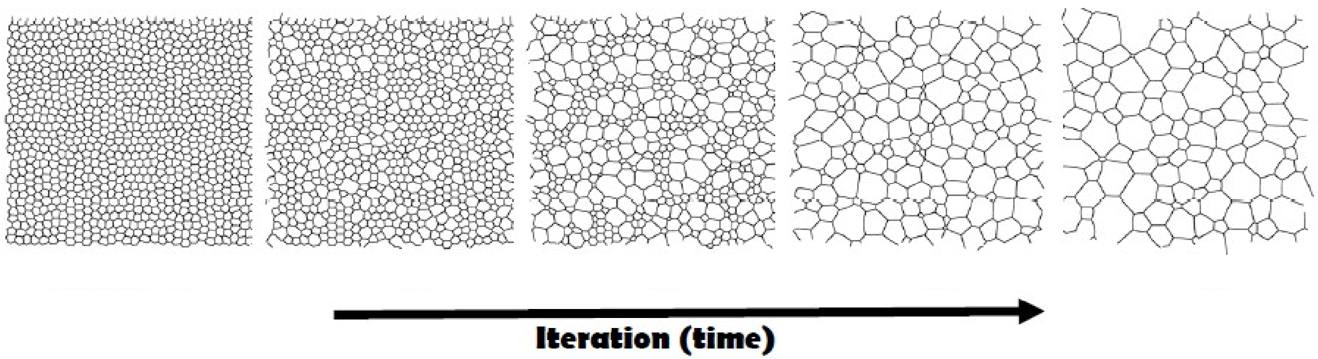
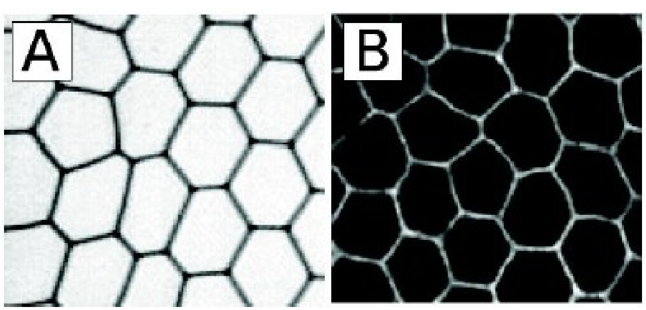



**- Study of biological tissues :** *some tissues get ordered, some others get disordered during development*



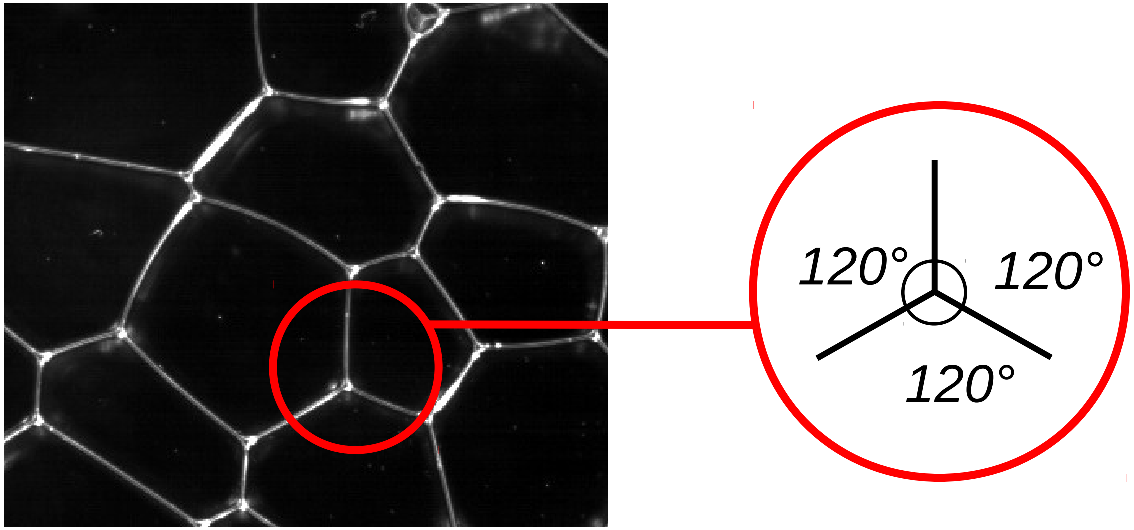

*A : 2D foam (M. Asipauskas).*

*B : epithelial tissue, Blankenship et al. (2006).* *credits : J. Käfer*

# Metastable states of a 2D foam

## 1/ Physical constraints (balance of forces)

- Plateau's laws :



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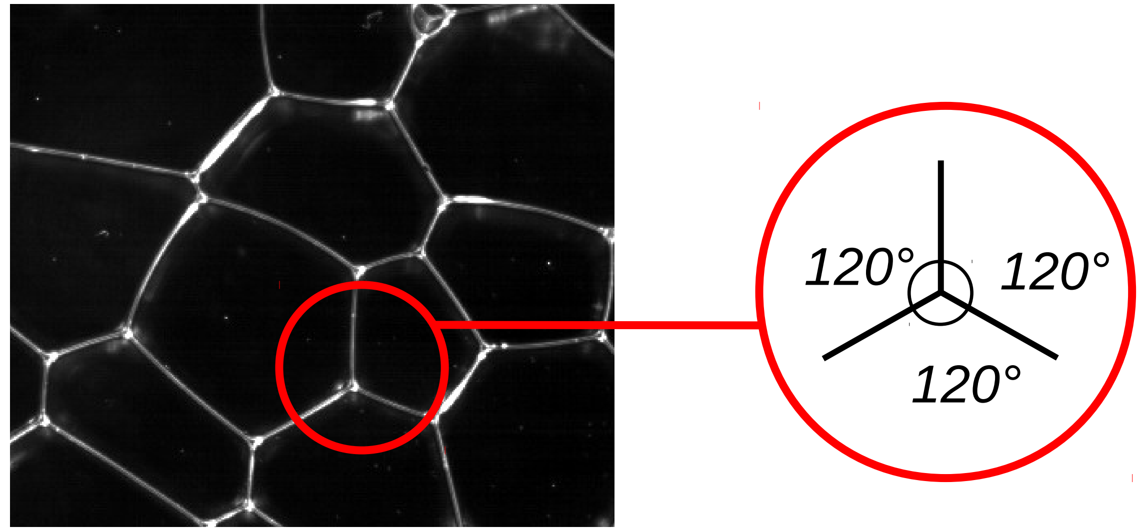


- Laplace's law : $\kappa_{ij} = (\Pi_j - \Pi_i)/\gamma$

*pressures*

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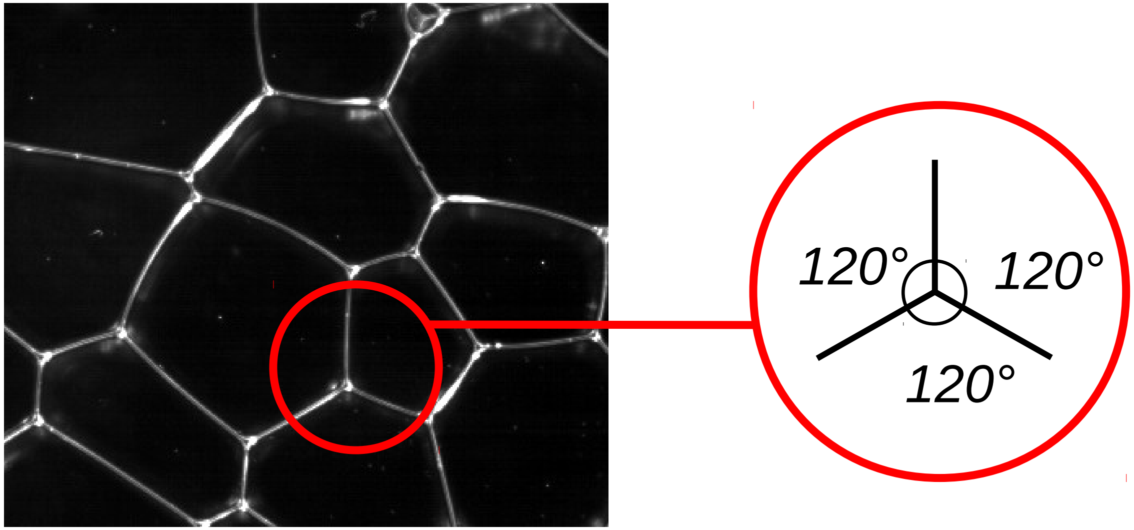


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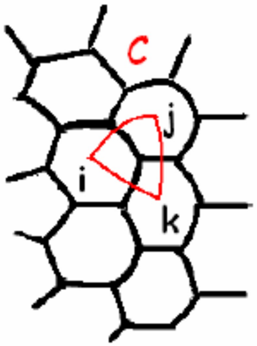
$$\kappa_{ij} + \kappa_{jk} + \kappa_{ki} = 0$$

*« curvature sum rule »*



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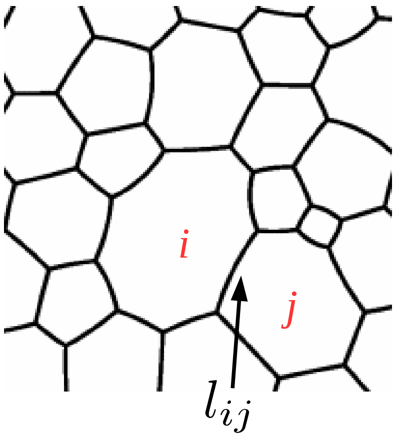
$$\sum_{j=1}^{n_i} l_{ij}\kappa_{ij} = \frac{\pi}{3}(n_i - 6)$$

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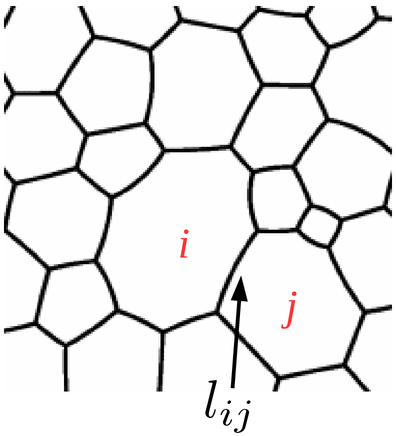


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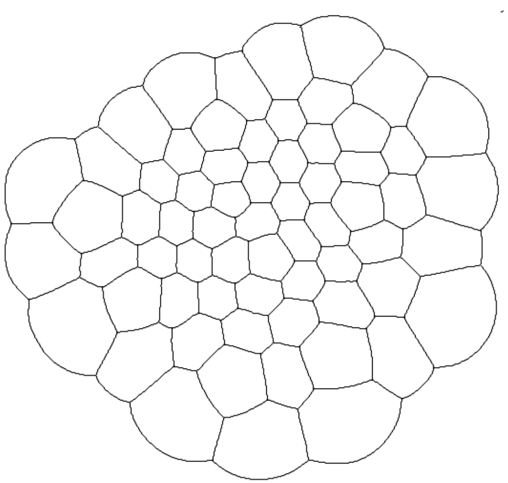
- Euler formula:

$$N_B + N_v - N_e = c$$

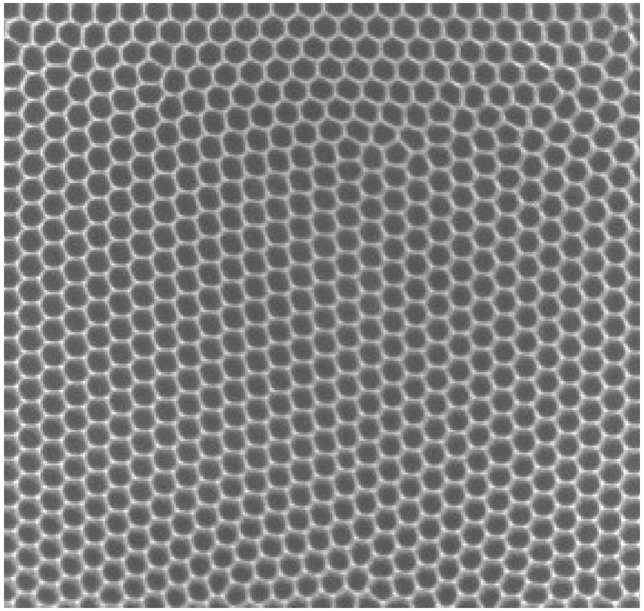


$N_B, N_v, N_e$ : *# of bubbles, vertice, and edges*

$c$ : *Euler-Poincaré characteristic* ~ 1

note : Plateau + Euler $\xrightarrow{N_B \gg 1}$ $\langle n \rangle = 6$

# Evidence of correlation between geometry (size) and topology (number of sides)



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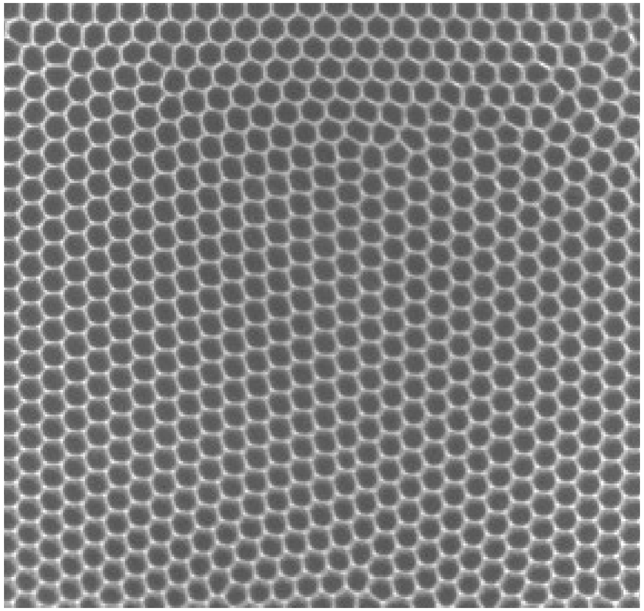

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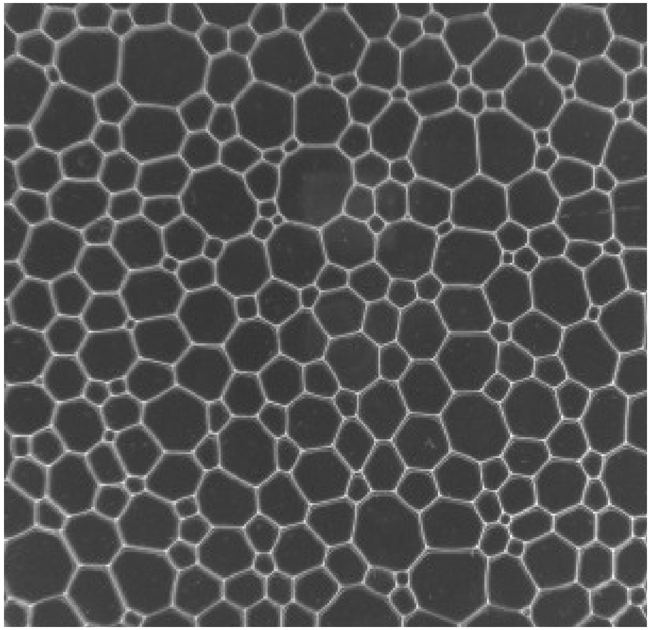

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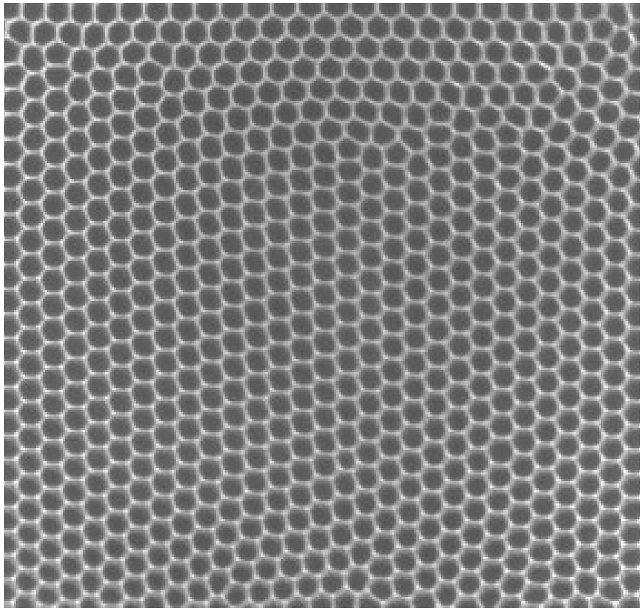

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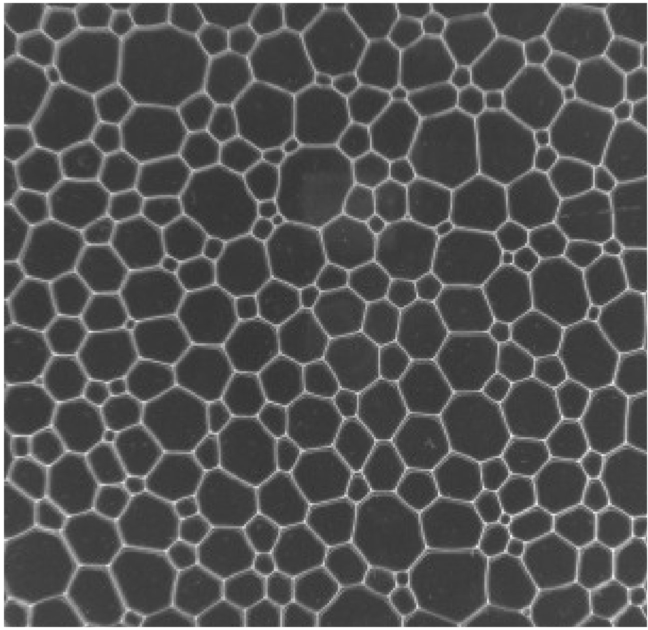

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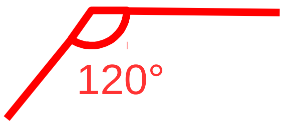

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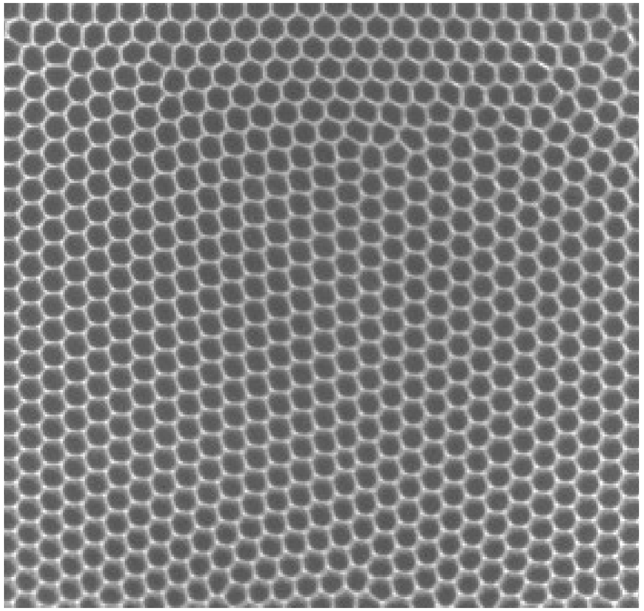

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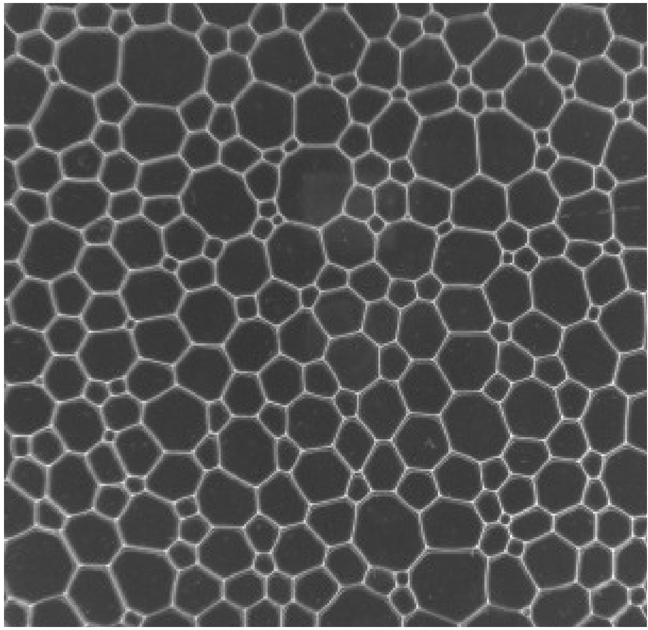

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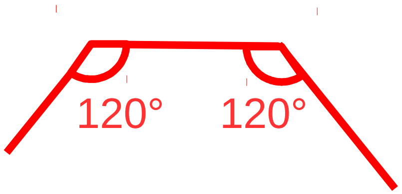

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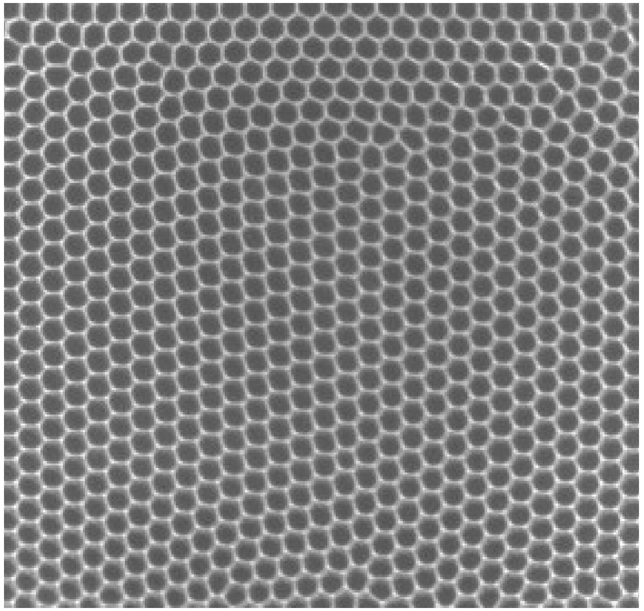

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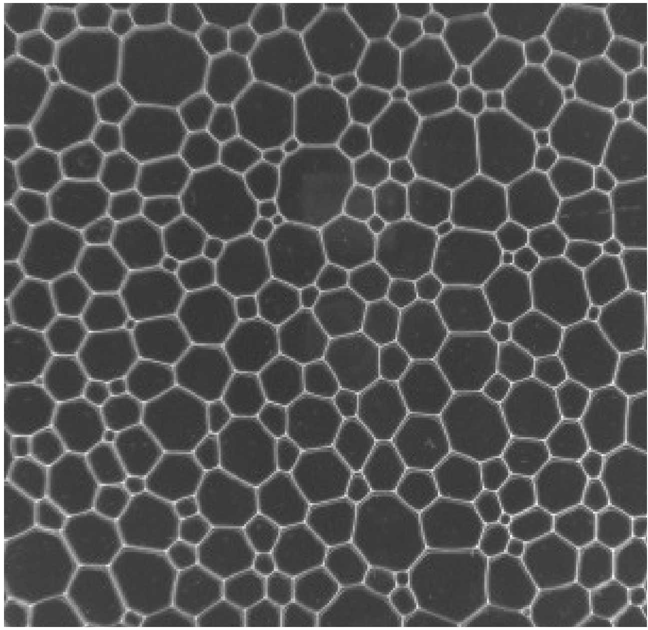

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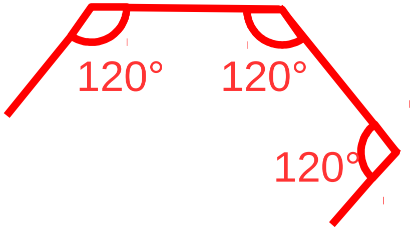
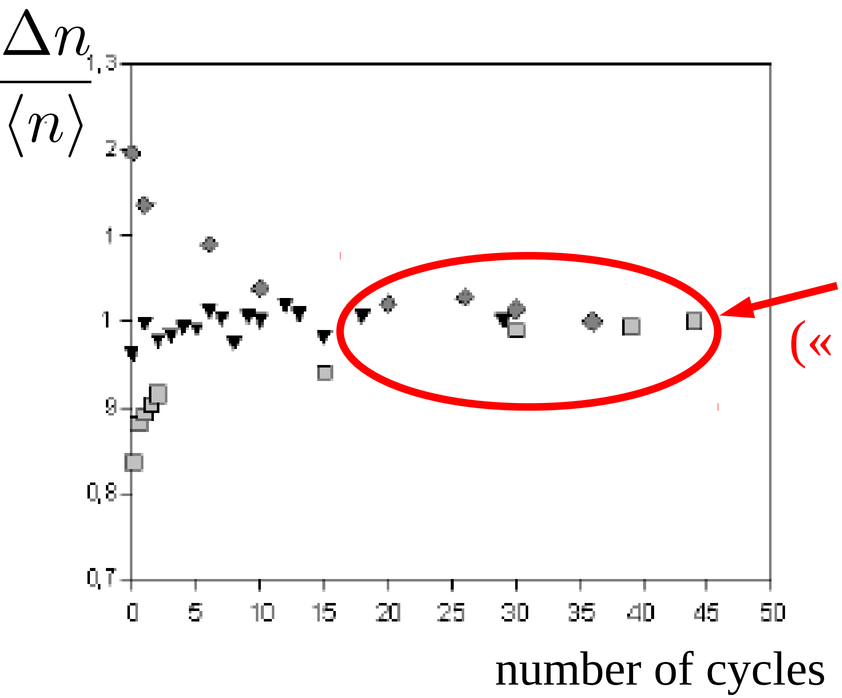
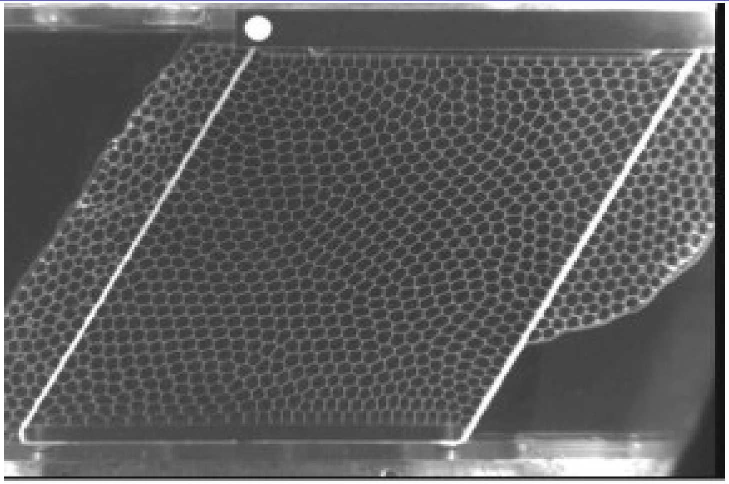


# Topo-geo correlation in a shuffled foam

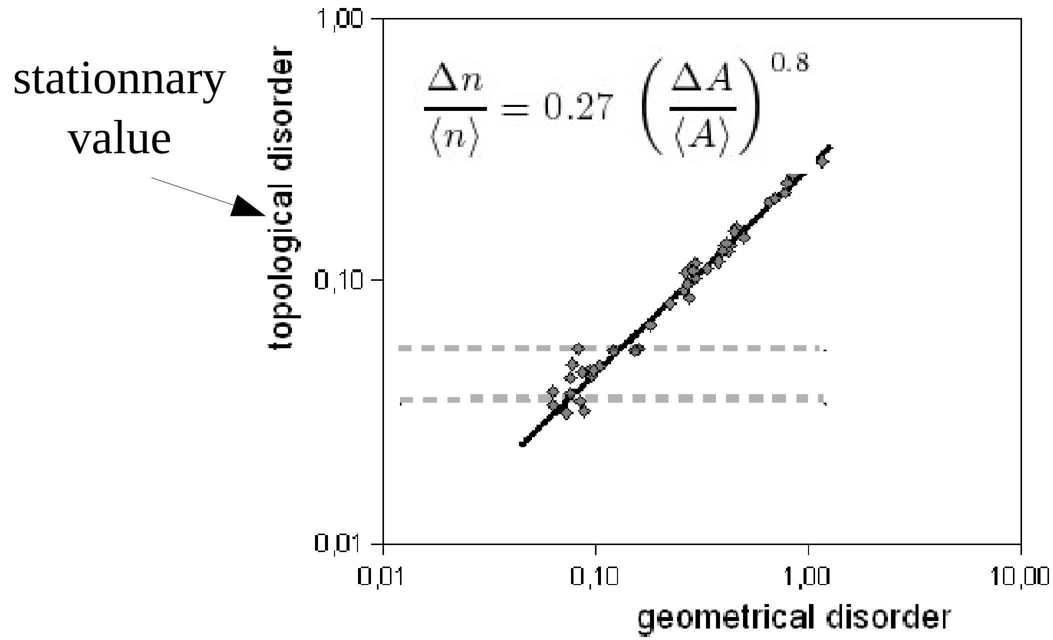
$p(A)$ (and so $\dfrac{\Delta A}{\langle A \rangle}$ ) is fixed.



C. Quilliet et al. (2008).

Tends to a stationnary value
(« thermodynamic » equilibrium ?)



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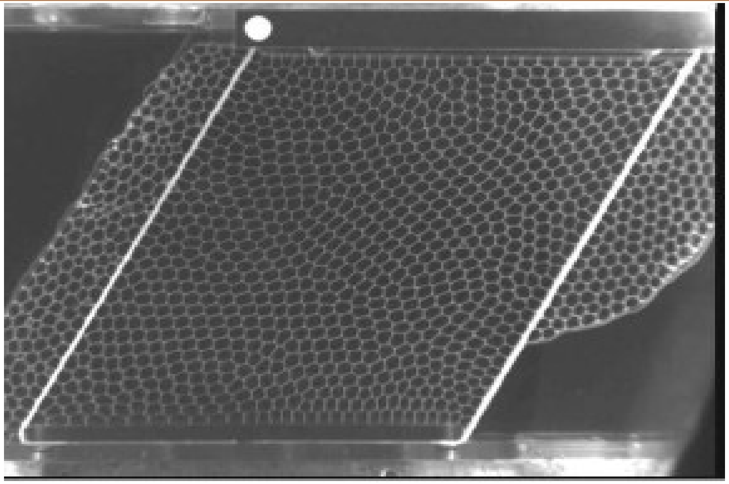

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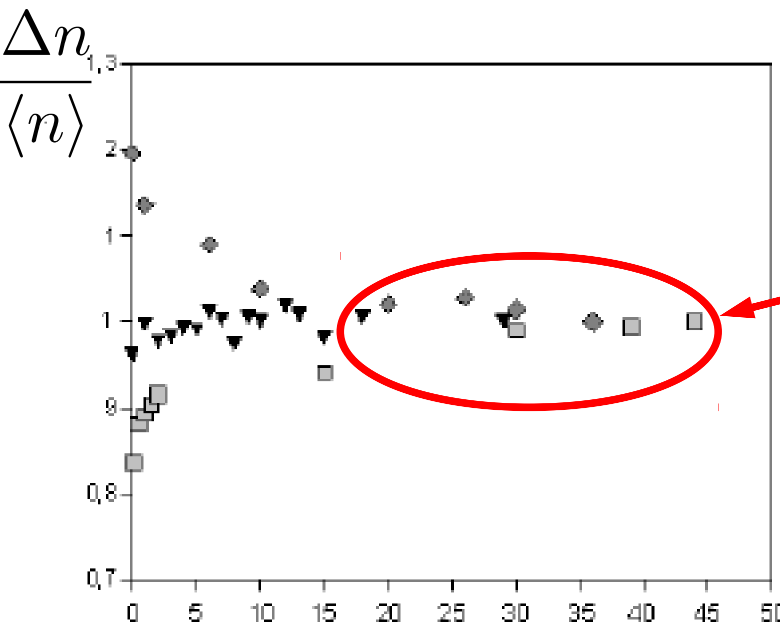


number of cycles

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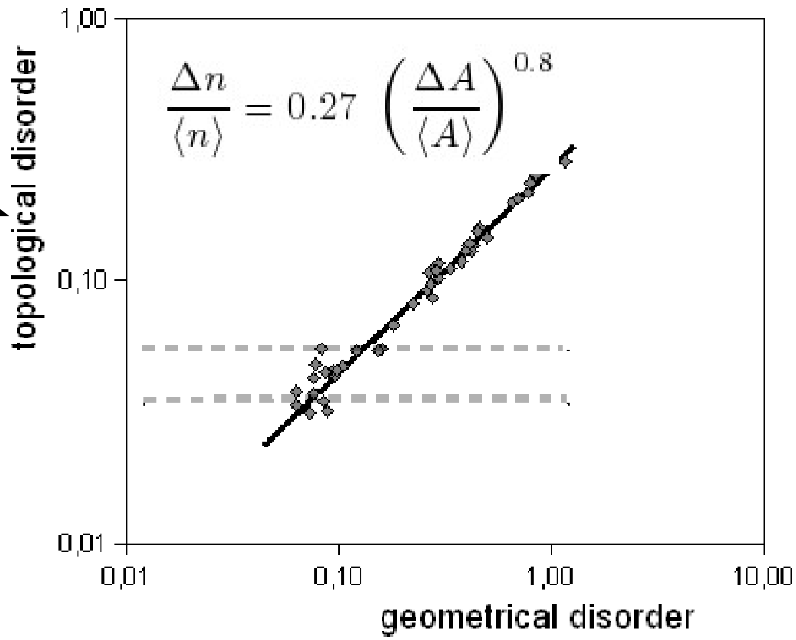
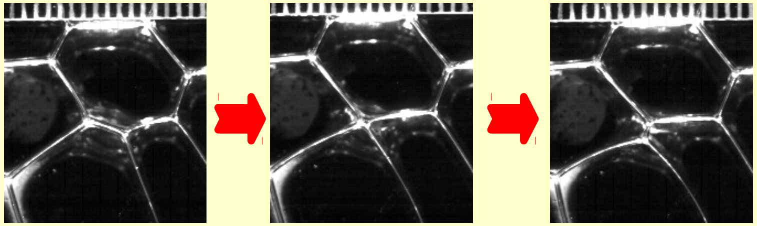
stationnary value




Topological T1 event :

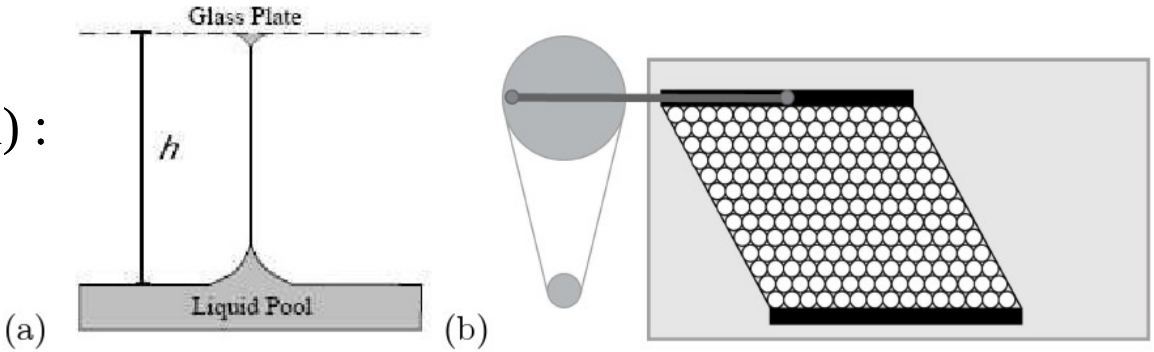
*stable* *unstable* *stable*



Can we use formalism of statistical physics to describe a shuffled foam ?

# Different kinds of shuffling

-Experiences (C. Quilliet & S. Ataei Talebi) :



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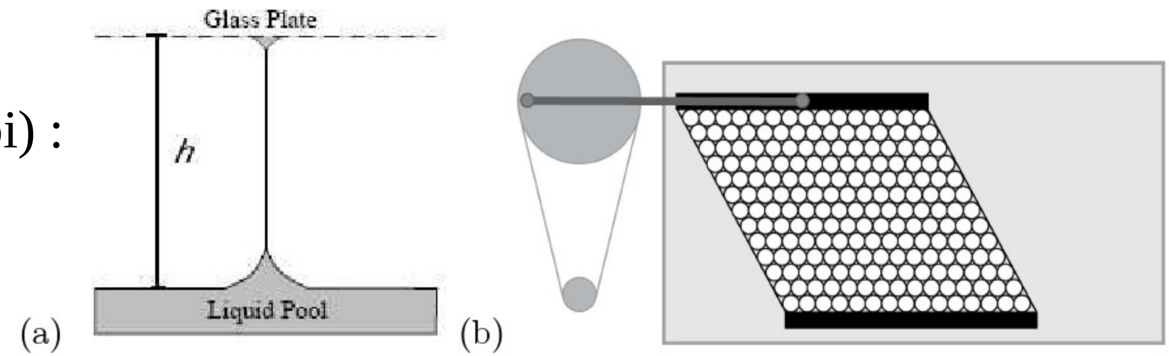
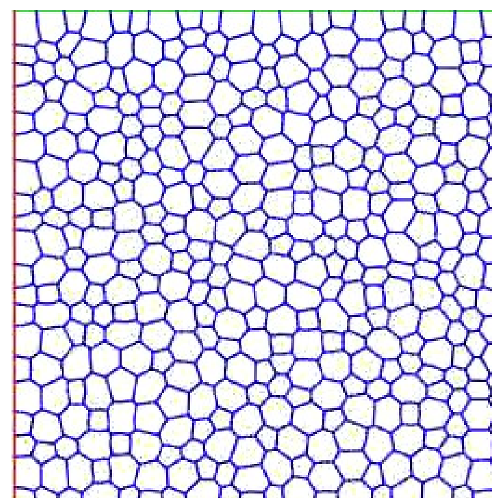



-Surface Evolver (S. Cox) :



out-of-equilibrium systems

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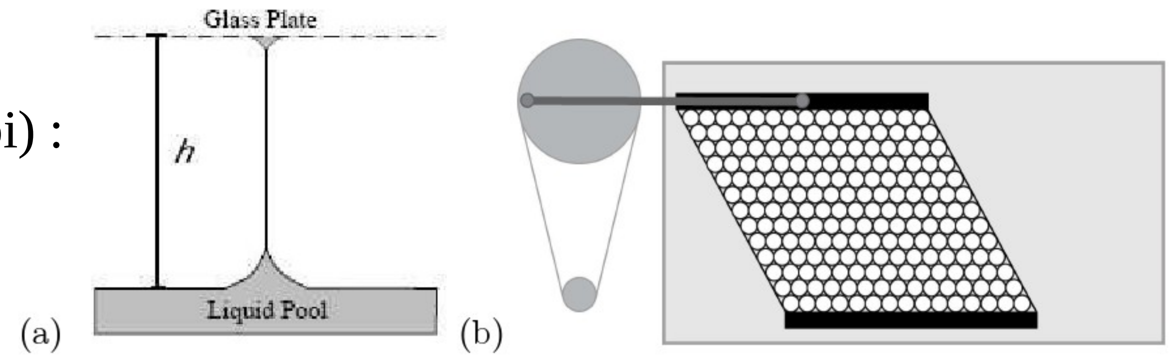


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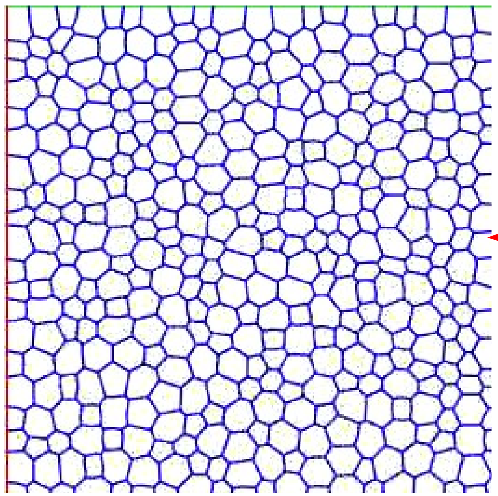
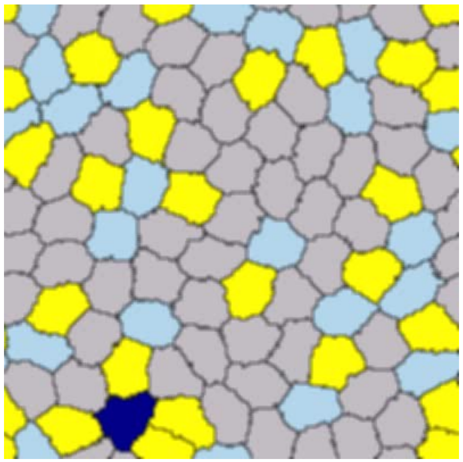


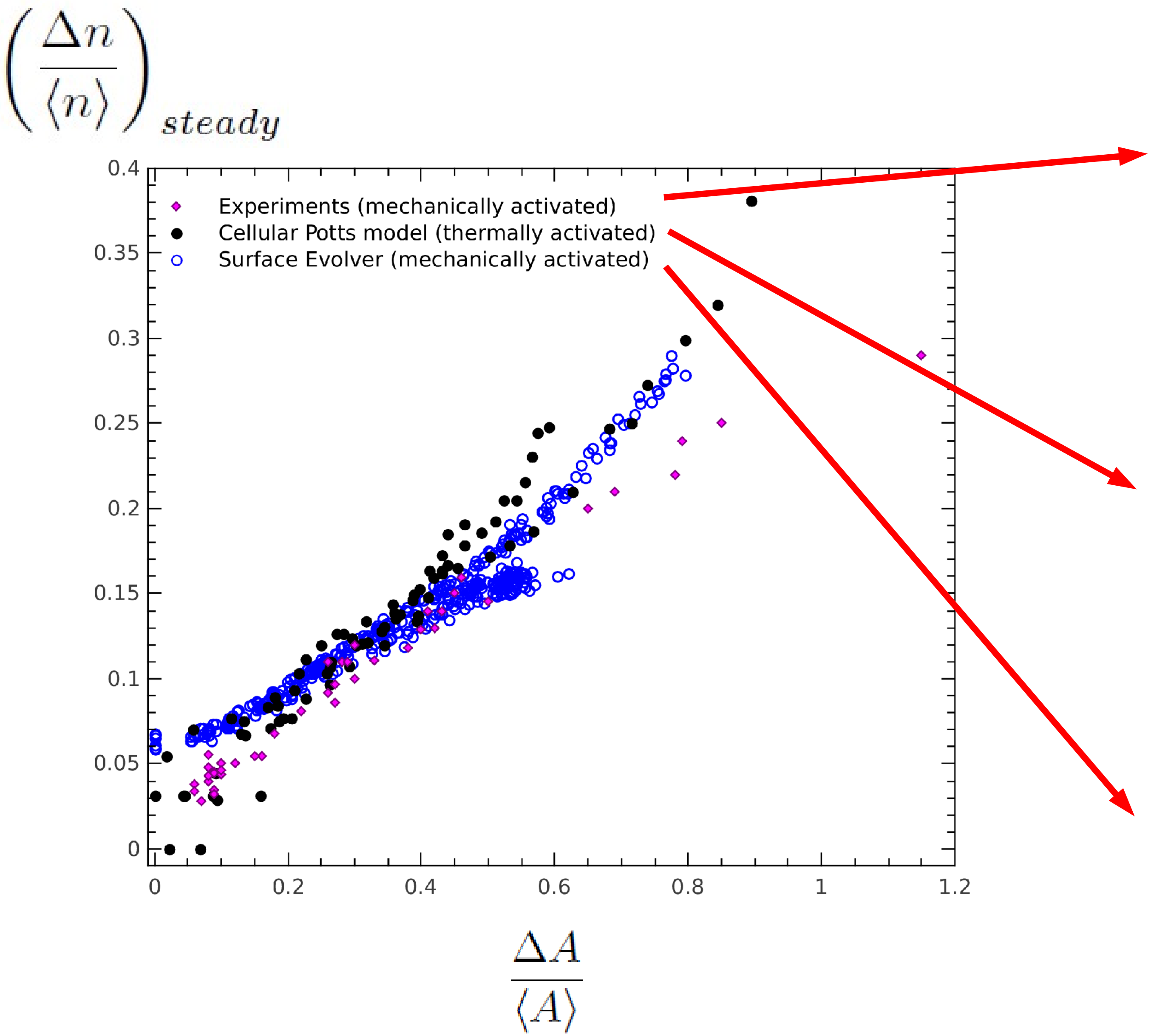
out-of-equilibrium systems

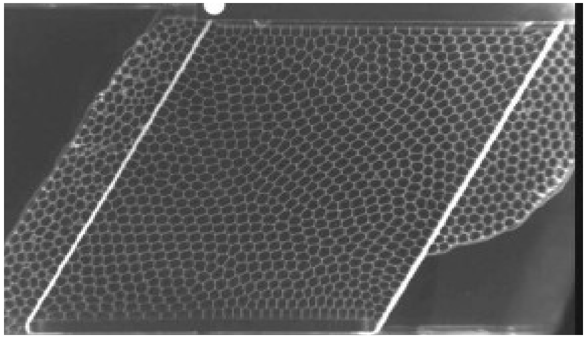
-Cellular Potts model (J. Käfer) :

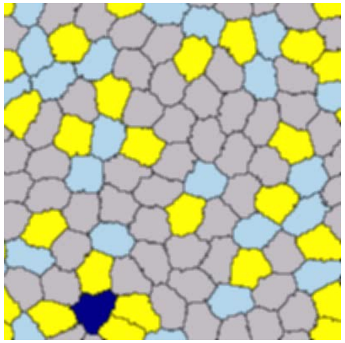


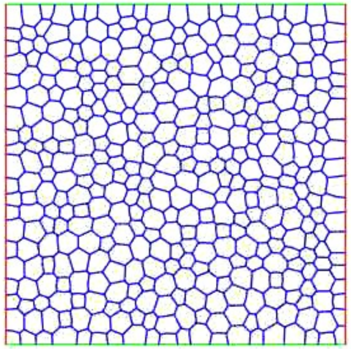
system at thermal equilibrium

- Experimental and numerical evidence that $\left(\frac{\Delta n}{\langle n \rangle}\right)_{steady}$ and $\frac{\Delta A}{\langle A \rangle}$ are **correlated :**







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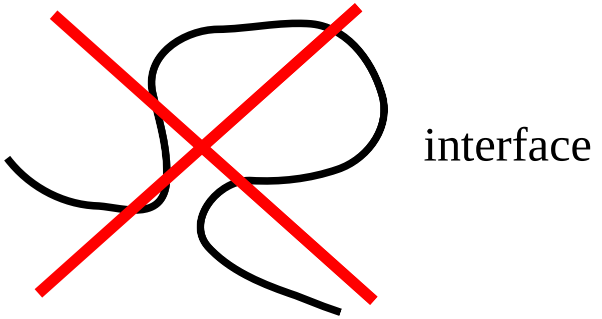

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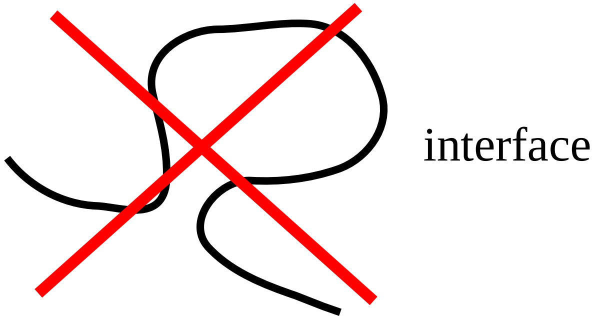


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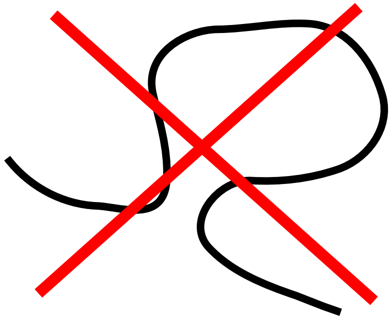

interface

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**Postulate : all the accessible states are visited with same probability**

# Model: Mean Field Approximation

- Consider one specific bubble, of given size $A_i$ and side number $n_i$

⇨ consists in neglecting the fluctuations of its neighborhood

⇨ ≡ bubble surrounded by a uniform foam (averaging over other bubble positions)

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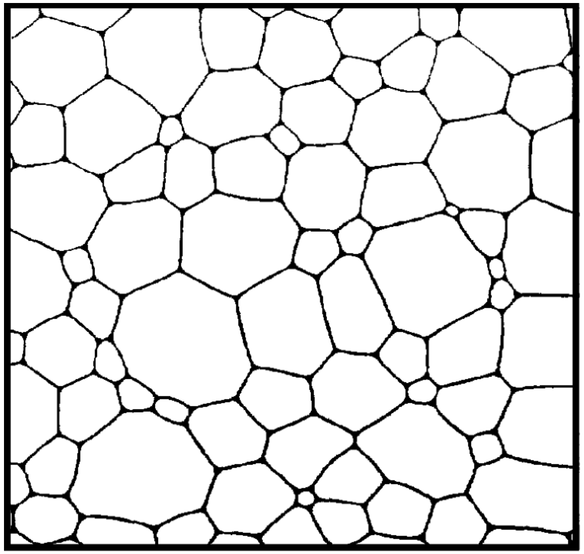
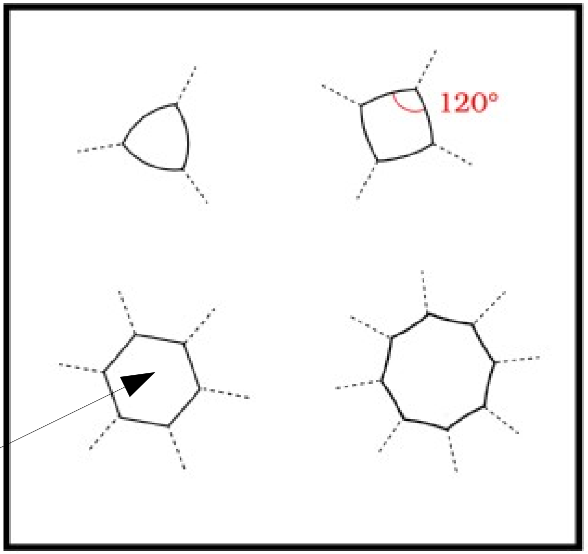
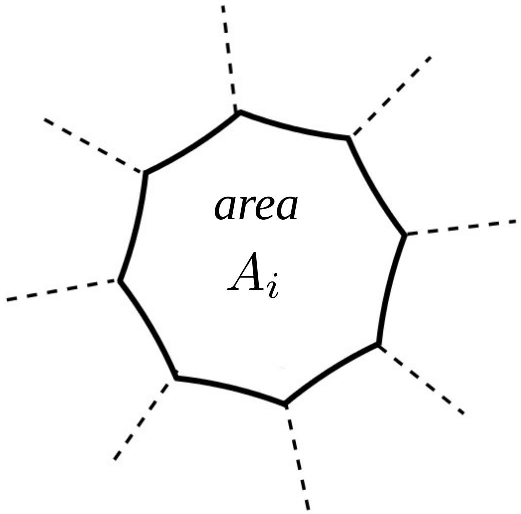
- Assuming homogeneous and isotropic foam, bubbles have regular geometries






**regular cells** : respect all geometrical, topological, and physical constraints, except Laplace's law

# Mean Field Approximation

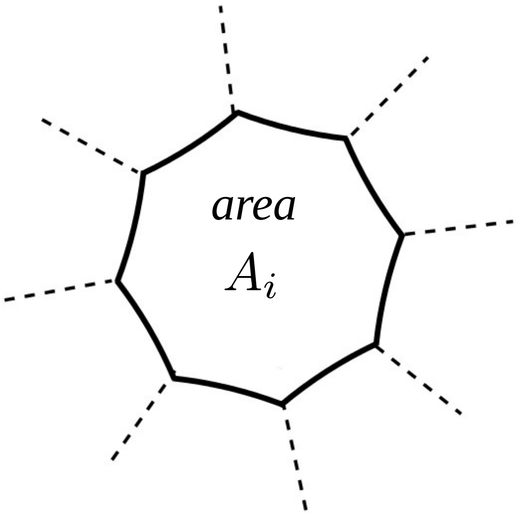



pressure drop : $\Pi_i - \Pi_j \simeq \langle \Pi_i - \Pi_j \rangle_{\{L\} \to n_i} = \gamma \frac{\pi}{3} \frac{(n_i - 6)}{e\sqrt{A_i}}$

*depends on $A_i$ and $n_i$ only*

$e \simeq 3.72$

# Mean Field Approximation

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*depends on $A_i$ and $n_i$ only*

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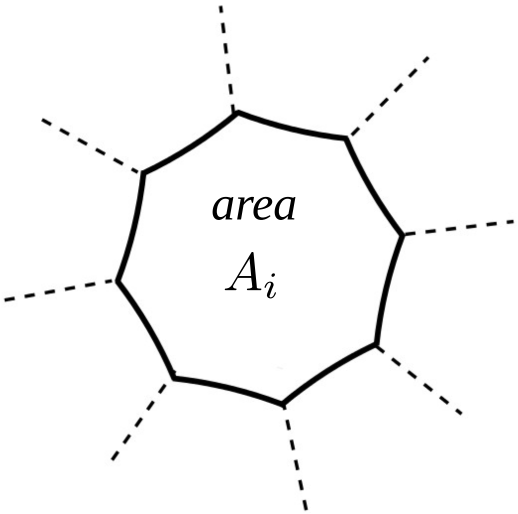


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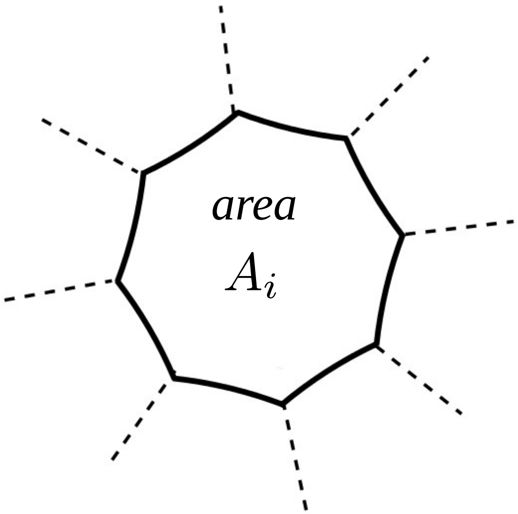


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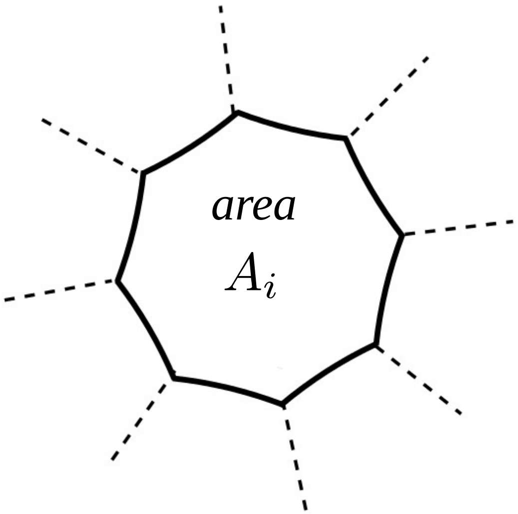


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$\Rightarrow \kappa_{tot} = \sum_i \kappa_i = 0$

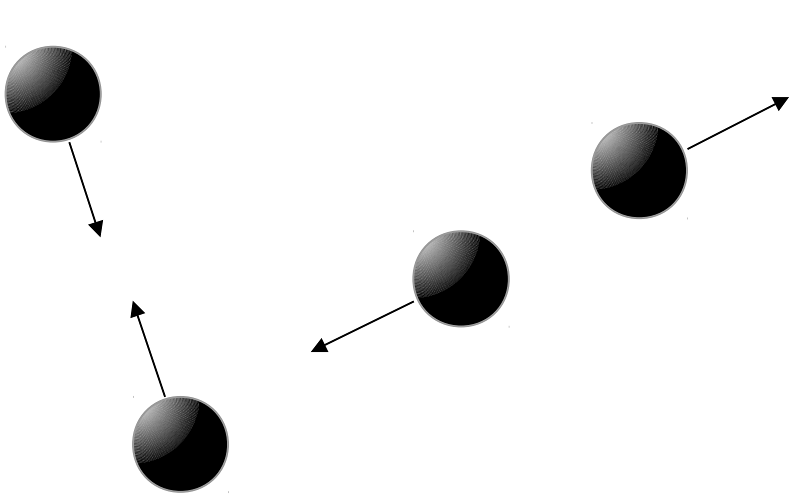
**& Euler formula :** $\sum_{i=1}^{N_B} n_i = 6N_B$

for a very large foam

# Interpretation: collisionnal invariants

**Ideal gas**

collision

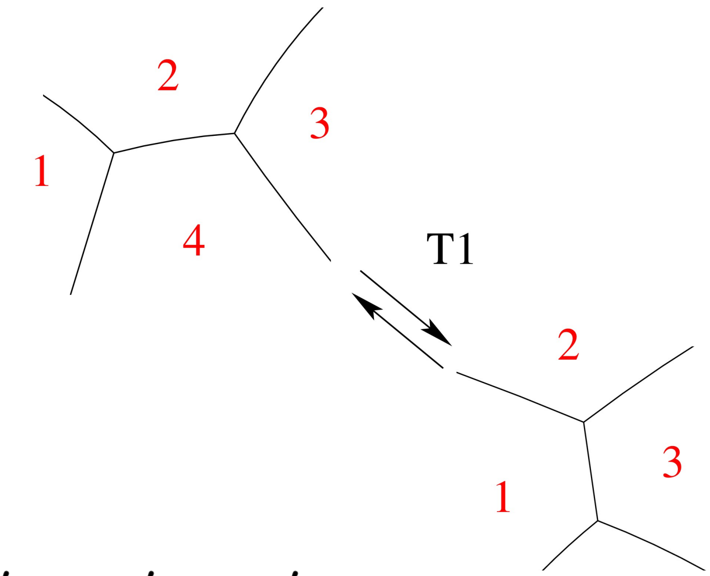


$$E_1' + E_2' = E_1 + E_2$$

$$\mathbf{p}_1' + \boldsymbol{p}_2' = \boldsymbol{p}_1 + \boldsymbol{p}_2$$

**2D foam**

T1event




$$n_1' + n_2' + n_3' + n_4' = n_1 + n_2 + n_3 + n_4$$

$$\kappa_1' + \kappa_2' + \kappa_3' + \kappa_4' = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$$

$n_i$ : Number of sides of bubble *i*

$\kappa_i = \sum_j \kappa_{ij}$ : Cell curvature of bubble *i*

# « Grand-canonical » description

Consider 1 cell of size *A*. **Rest of the foam = reservoir of sides and curvature**

→ **Probability for a cell with given area A to have *n* sides :**

$$p_A(n) \propto \exp\left(-0.28\beta\frac{n(n-6)}{\sqrt{A}} + \mu n\right)$$

$\beta^{-1},\ \mu\beta^{-1}$ : *effective « temperature » and « chemical potential ».*

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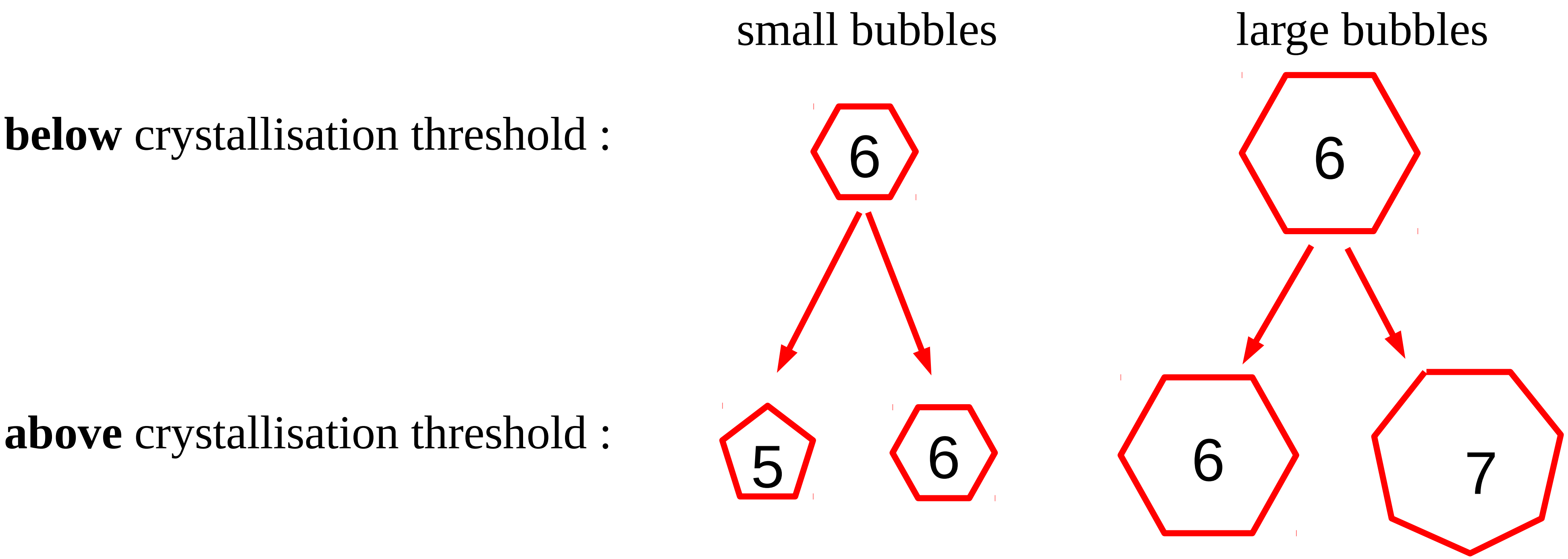
**No free parameter :** $\beta$ and $\mu$ obtained (numerically) by solving :

$$\langle n \rangle = 6, \qquad \left\langle \frac{\gamma\pi}{3}\frac{n(n-6)}{e\sqrt{A}} \right\rangle = 0$$

M. Durand, *EPL* 90, 60002 (2010).

# At low dispersities: order-disorder transition

e.g. : **bidisperse foams** → 2 parameters : $\alpha = \frac{N_{small}}{N_{small} + N_{large}}$ , $r = \frac{A_{large}}{A_{small}}$




M. Durand & *al.*, to appear in *PRE*.

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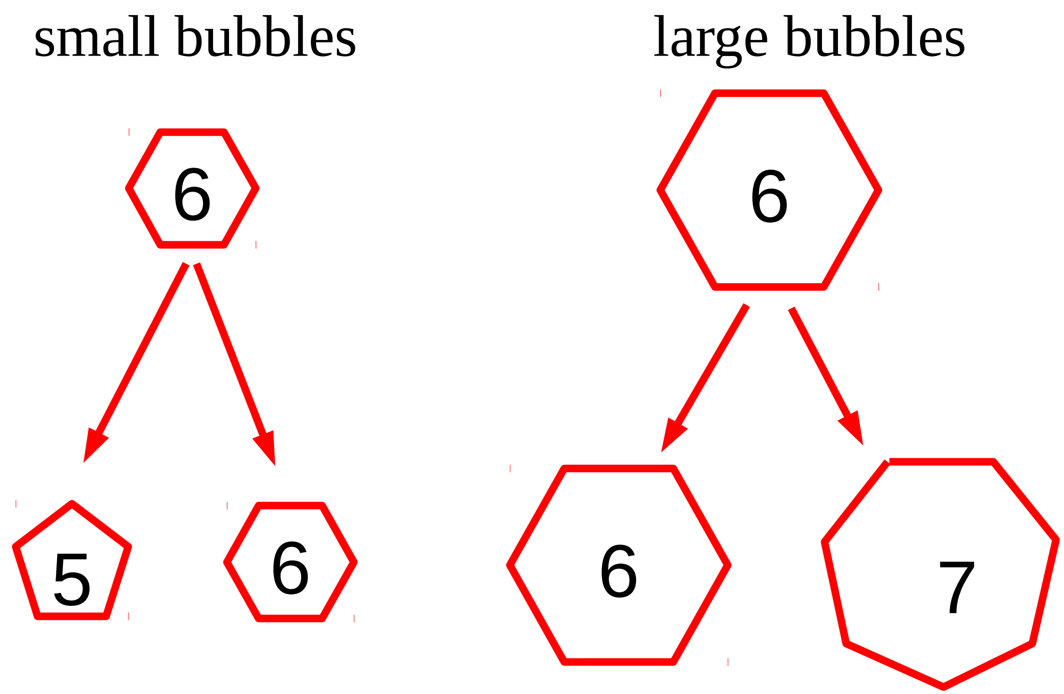
**below** crystallisation threshold :

**above** crystallisation threshold :




**size ratio threshold :** $r_c = \frac{A_{large}}{A_{small}} = \left(\frac{7}{5}\right)^2 = 1.96$

M. Durand & *al.*, to appear in *PRE*.

# At larger dispersities: analytical approximations

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**Analytical approximations :**

$$\beta^{-1} \simeq 5.06 \frac{\langle A^{1/2}\rangle\langle A^{-1/2}\rangle - 1}{\langle A^{1/2}\rangle}$$

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→ related to the **width** and **mean** of the size distribution

**Topological-geometrical correlations :**

- at the level of **individual bubbles** :
(mean number of sides of a cell of size *A*)

$$\bar{n}(A) = 3\left(1 + \frac{\sqrt{A}}{\langle\sqrt{A}\rangle}\right)$$

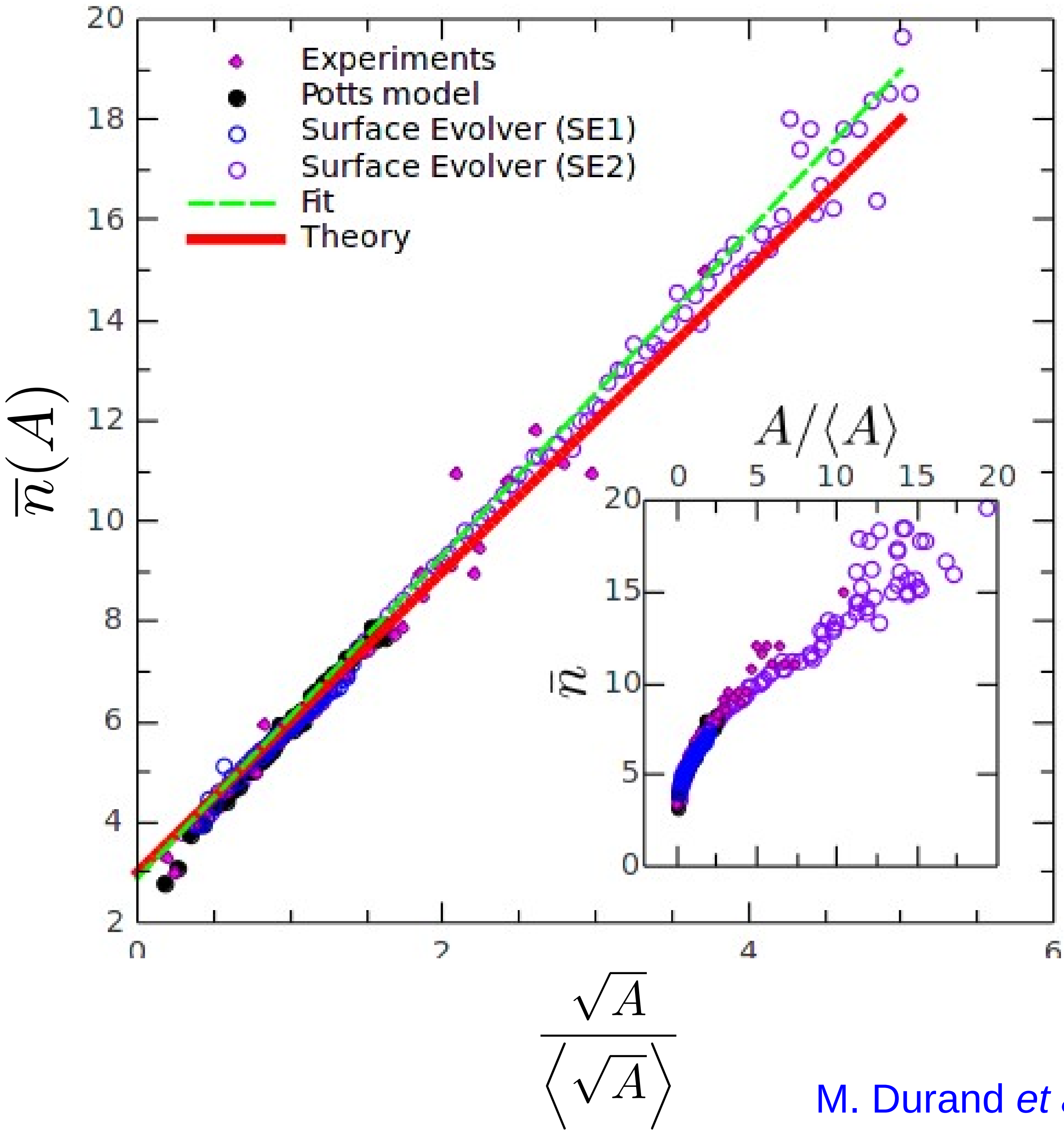
- at the level of the **entire foam** :

*topology* →

$$\left(\frac{\Delta n}{\langle n\rangle}\right)^2 = \frac{1}{4}\left(\langle A^{1/2}\rangle\langle A^{-1/2}\rangle + \langle A\rangle\langle A^{1/2}\rangle^{-2} - 2\right)$$

← *geometry*

**Topo-geo correlations at the level of individual bubbles :**

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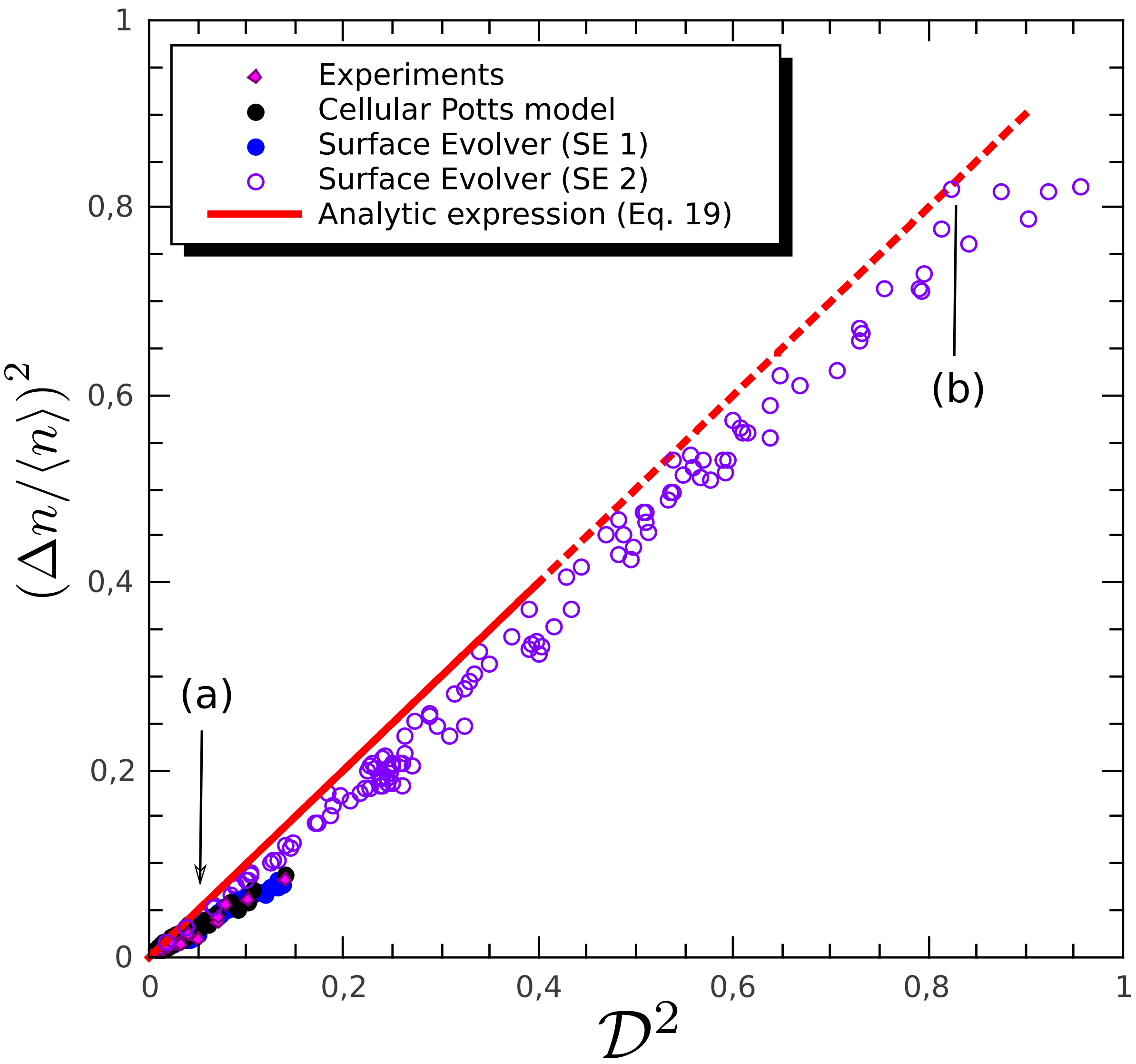
M. Durand *et al.*, *PRL* 107, 168304 (2011).

# Topo-geo correlations at the level of the entire foam :




$$\left(\frac{\Delta n}{\langle n\rangle}\right)^2 = \mathcal{D}^2$$

$$4\mathcal{D}^2 = \langle A^{1/2}\rangle\langle A^{-1/2}\rangle + \langle A\rangle\langle A^{1/2}\rangle^{-2} - 2$$

# Conclusions

- topological-geometrical correlations in shuffled foams, independent of the nature of shuffling (to some extent...)

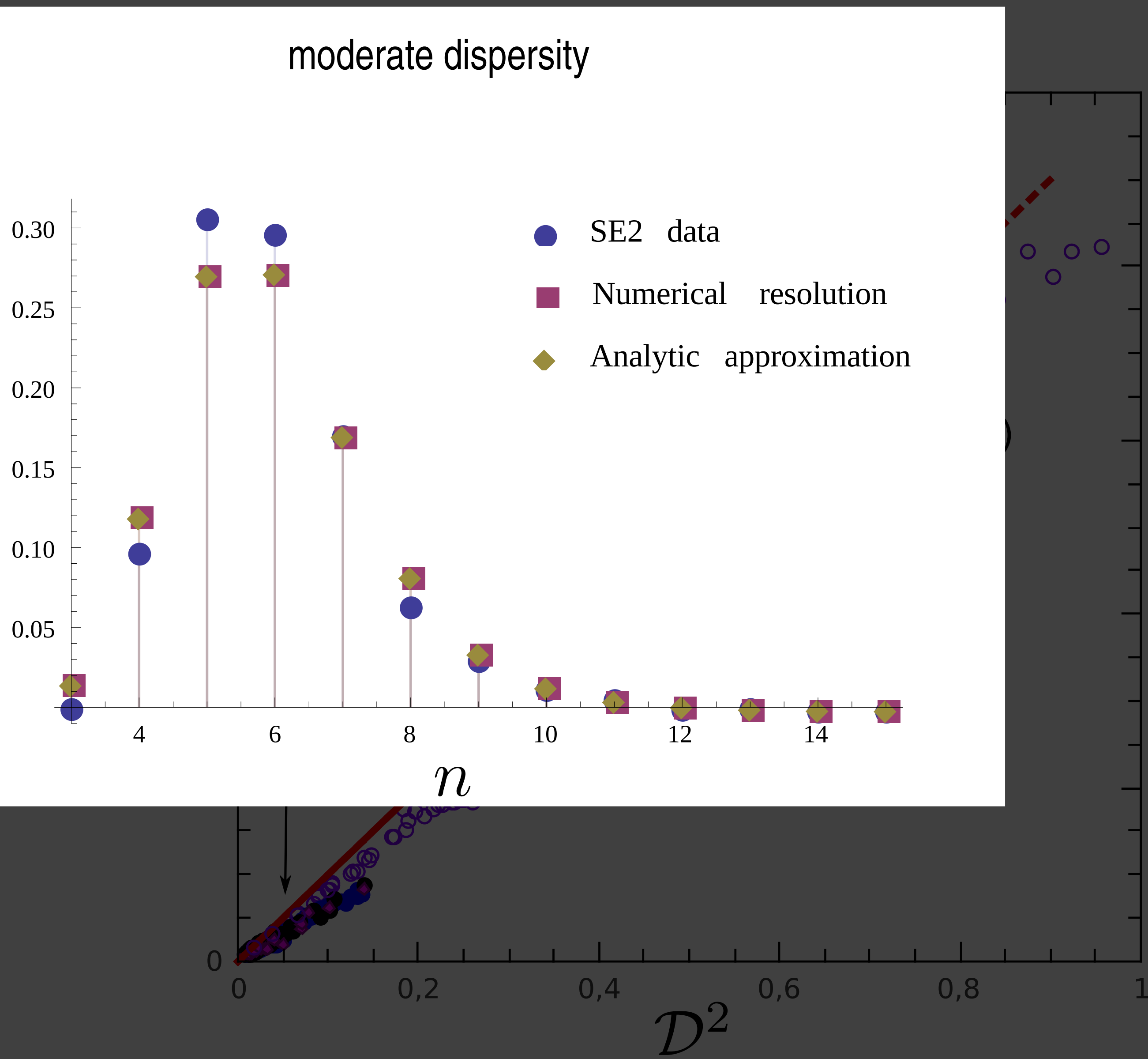
- Universal relationship between $\frac{\Delta n}{\langle n \rangle}$ and $\mathcal{D}$

- Mean-field theory that neglects correlations between neighours, but keeps correlations between *A* and *n*.

- Prediction of a topological order-disorder transition (not yet observed/studied?)

Thanks !

Topo-geo correlations at the level of the entire foam :

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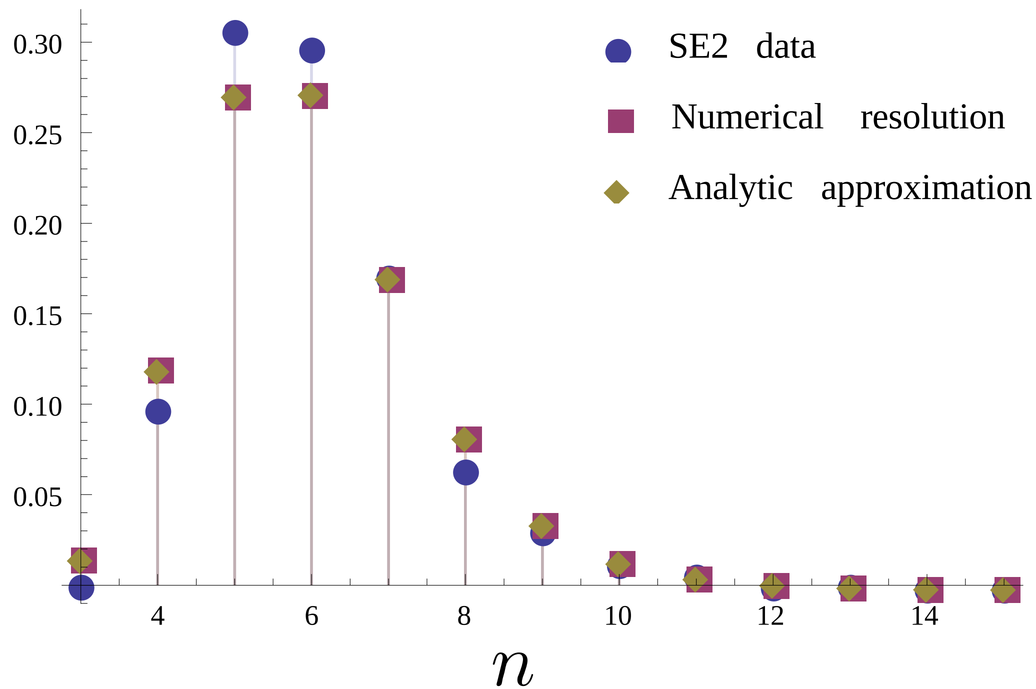
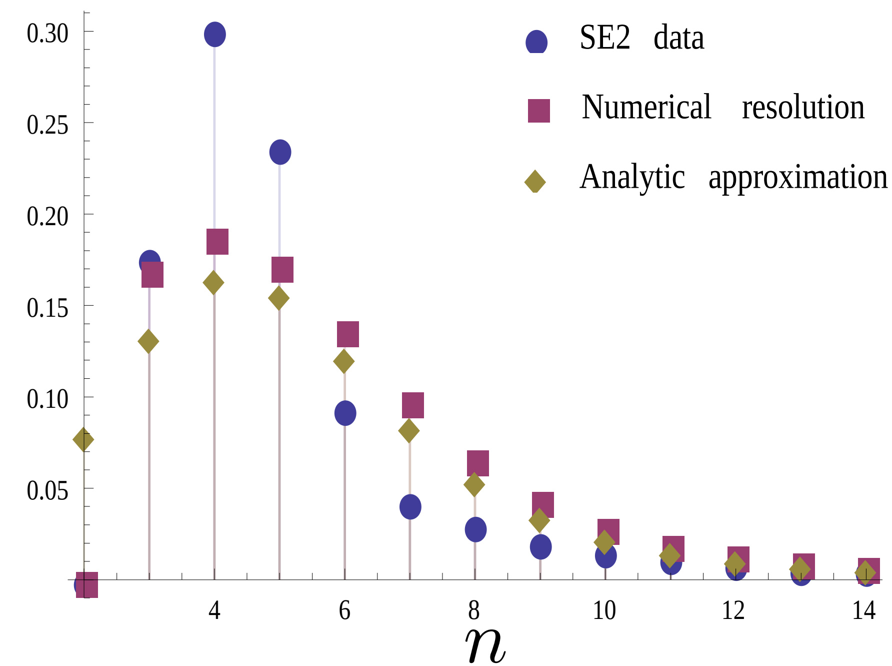
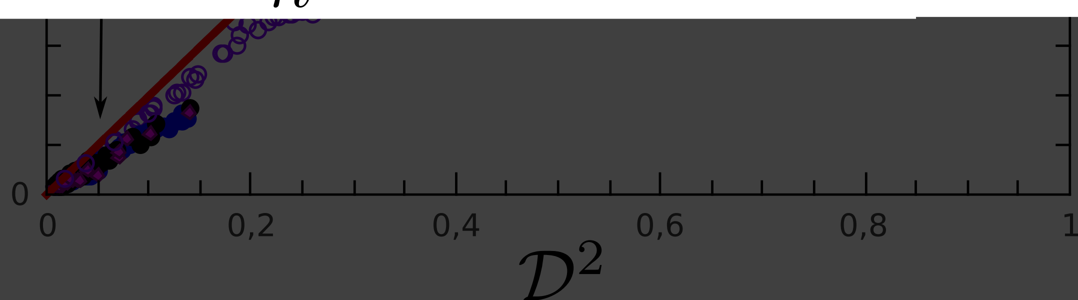
## Topo-geo correlations at the level of the entire foam :

moderate dispersity




(very) large dispersity



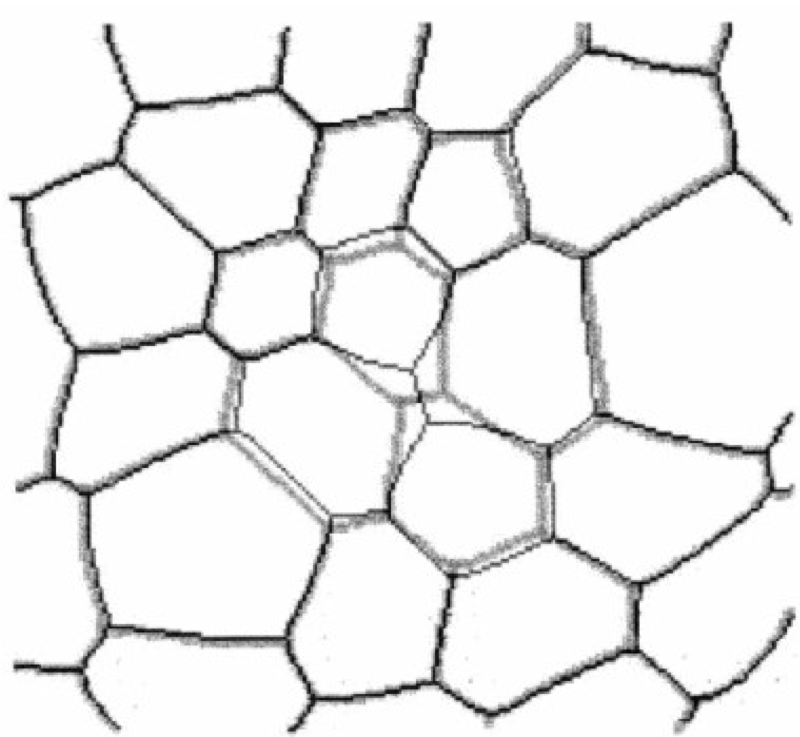




$$4\mathcal{D}^2 = \langle A^{1/2}\rangle\langle A^{-1/2}\rangle + \langle A\rangle\langle A^{1/2}\rangle^{-2} - 2$$

Note : **short-range correlations** between bubbles :

- correlation length ~ few bubble radii (Durian et al., *Science* 1991, Duri et al., *PRL* 2009)

- superposition of the metastable states before and after a T1 event (F. Elias et al. 1999) :



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See Rivier & coworkers, Fortes & Texeira, Almeida & Iglesias, Sire & Seul, Godrèche & al.

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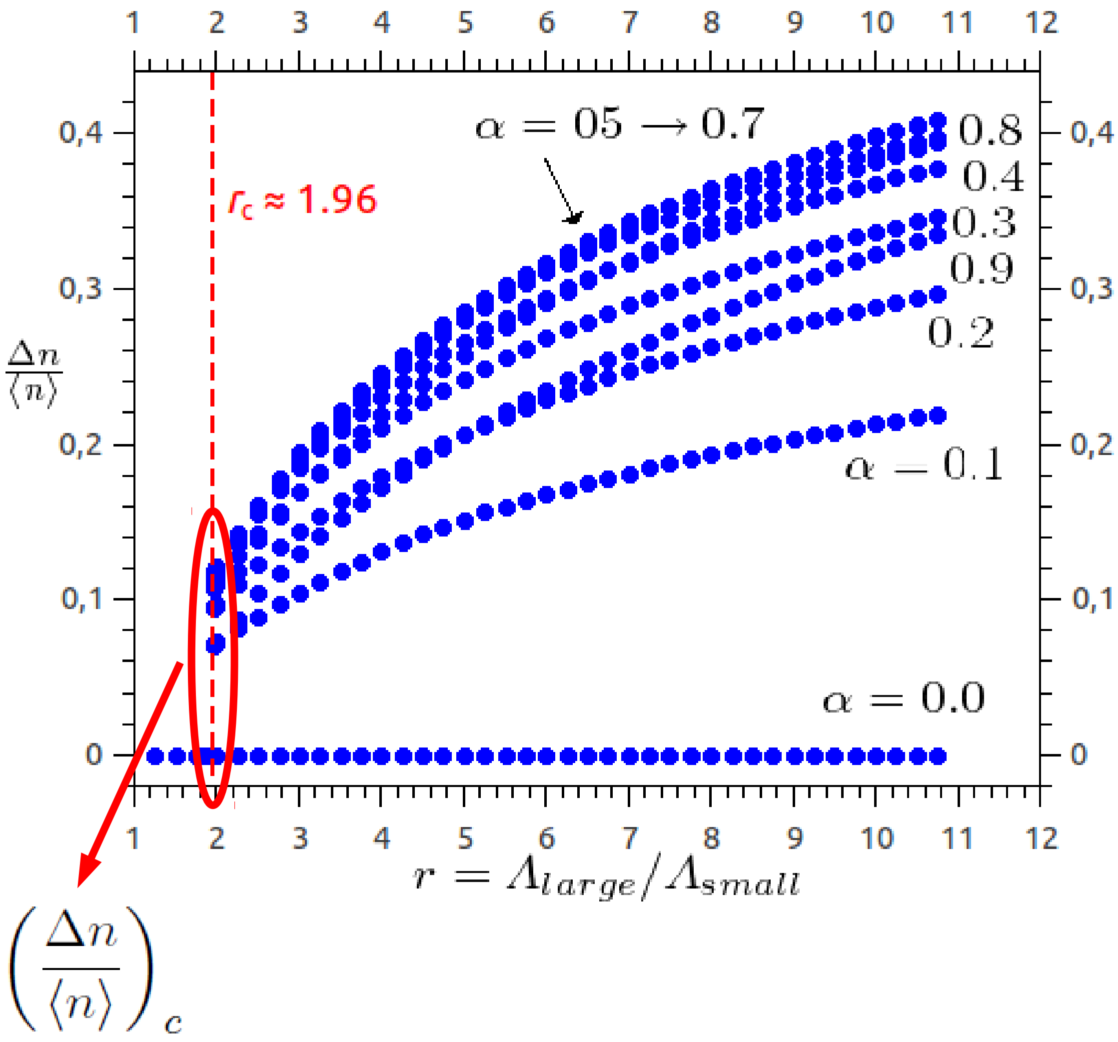
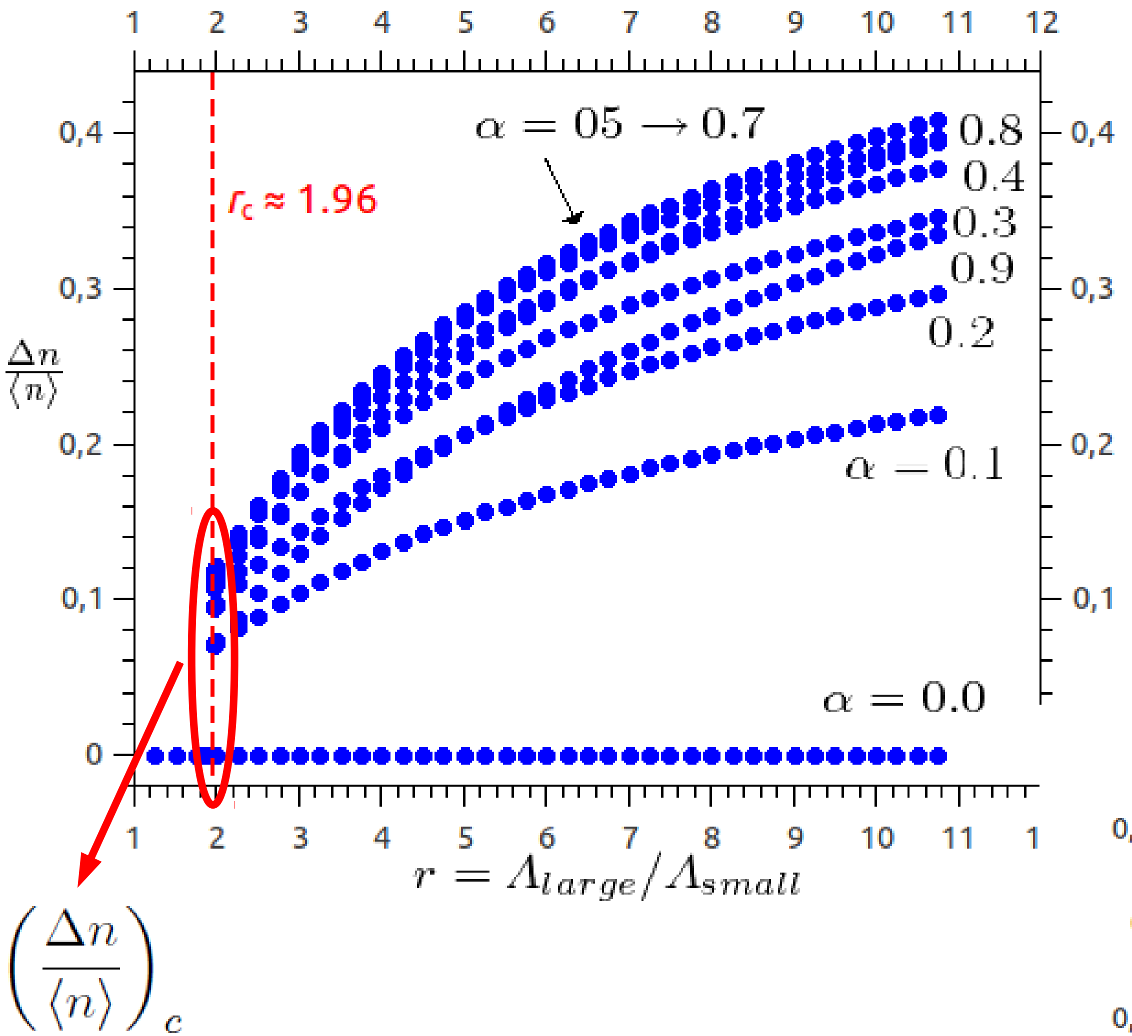
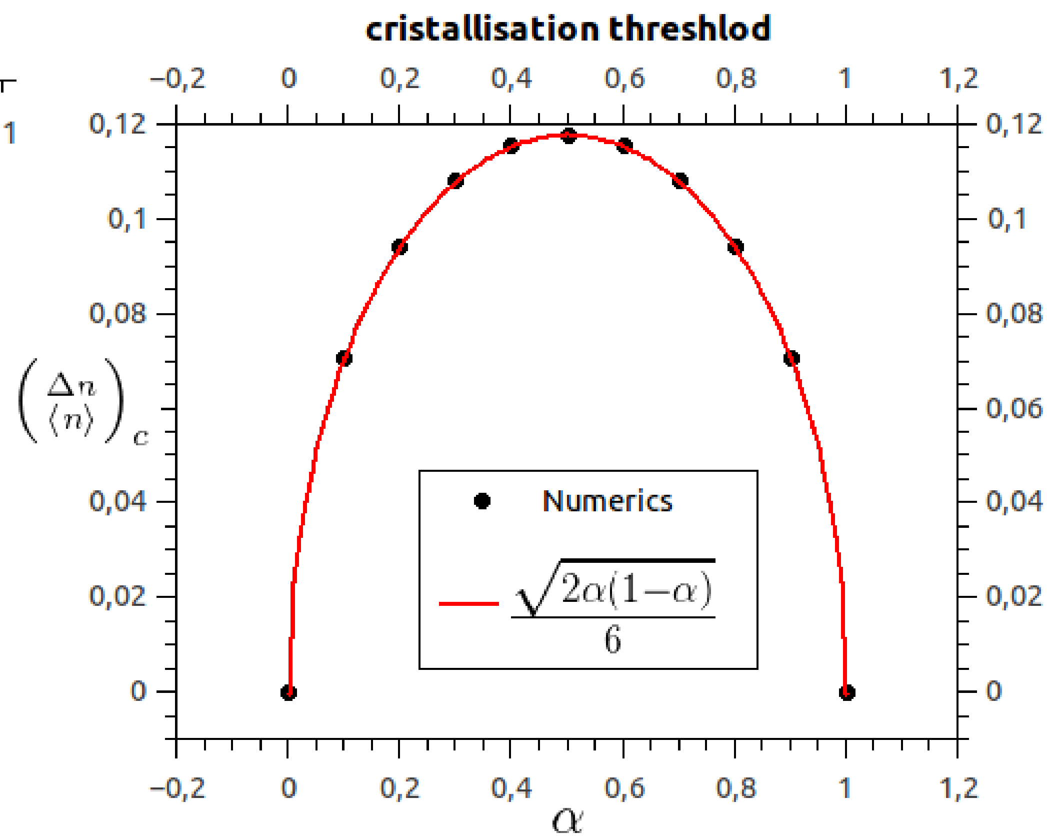
**BUT**, **either** focused on topological description only, **or** bubble areas treated as random variables (subjected to some constraints)

**HERE**, number and sizes of the bubbles are **fixed** ( and so is $p(A)$ ).



$\alpha = 05 \rightarrow 0.7$
$r_c \approx 1.96$
0.8
0.4
0.3
0.9
0.2
$\alpha - 0.1$
$\alpha = 0.0$
$\frac{\Delta n}{\langle n \rangle}$
$r = \Lambda_{large}/\Lambda_{small}$
$\left(\frac{\Delta n}{\langle n \rangle}\right)_c$
0,4
0,3
0,2
0,1
0
1
2
3
4
5
6
7
8
9
10
11
12




One can easily prove that :

$$\left(\frac{\Delta n}{\langle n \rangle}\right)_c = \frac{\sqrt{2\alpha(1-\alpha)}}{6}$$